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GREGORII FONTANAE

CLER. REG. SCHOL. PIAR.

IN REG. CAES. PAPIENSI UNIVERSITATE

SUBLIMIORIS MATHESEOS PUBLICI PROFESSORIS

DISQUISITIONES

PHYSICO-MATHEMATICAE

NUNC PRIMUM EDITAE.

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PAPIAE.

IN TYPOGRAPHEO MONAST. S. SALVATORIS.

PRAESID. REI LITTER. PERMITT.

ANNO MDCCLXXX.

1780
B.B.R.A.S.



FERDINANDO

HUNGARIAE ET BOHEMIAE PRINCIPI
AUSTRIAE ARCHIDUCI
INSUBRIAE AUSTRIACAE GUBERNATORI
ETC. ETC. ETC.



GREGORIUS FONTANA
FELICITATEM.

LUcubrationum Mearum, quas Tibi, FERDINANDE AUSTRIACE,

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nuncupo & voveo, non tam Te patronum ac vindicem opto, quam aeſtimatorem ac judicem vereor. Tuum ſcilicet iſtud ingenium peracre, acutiſſimum, ad omnia ſumma natura factum, incredibili celeritate praeter alia quamplurima Phyſicas etiam arripuit Mathematicaſque Diſciplinas, & pro indolis magnitudine vulgaria quaeque & in medio poſita deſpicientis in ſecretiora illarum adyta penetravit. Hinc

Tuus ille exſtitit ab incorrupto judicio nunquam diſjunctus amor, quo honeſtas artes ac Litteras ita complecteris, ut quantum iis, qui vere modeſteque ſapiunt, ultro lubenſque tribuis, tantum abhorreas a fucata iſta, grandiloqua & circulatoria philoſophia, quae aetatem noſtram bonorum aeque ac malorum feraciſſimam occupavit: ea enim eſt fere humanarum rerum conditio, ut quo tempore Scien-

tiarum ſtudia ardentiſſime excoluntur, & a ſummo perfectionis faſtigio videntur jam propius abeſſe, tunc praecipue inter paucos vere ſolideque ſapientes multae undique quaſi turmatim emergant cohortes ardelionum, qui magnis verbis ac mirificis promiſſis populo ſe venditantes viis omnibus ad famam graſſantur. Tu autem, PRINCEPS SAPIENTISSIME, *cum ab his Tibi imponi non ſinis, tum illos Tua quotidie con-*

ſummas benevolentia, aucto-ritate tueris, ornas benefi-ciis, nihil ut ſibi de Te polliceri non debeant Phi-loſophi ſancto hoc nomine digni, omneſque vel Tuo unius exemplo intelligere poſſint, quantum a Viris ip-ſis Principibus & populo-rum rectoribus ſincerae Vir-tuti ingenuiſque Artibus ſit tribuendum, quae ſane ſi magnis careant patrociniis, vel humiles concidant, vel debelitatae langueſcant, ne-

ceſſe eſt. Atque hunc Tuum effuſum in Litteras favorem ſenſit prae ceteris hoc noſtrum Papienſe Archigymnaſium , *quod Tuis auſpiciis paucorum annorum intervallo ad eam crevit amplitudinem gloriae, ut jam audacter poſſit cum Italicorum, atque adeo Europaeorum florentiſſimo quoque de principatu contendere. Quum enim his proximis annis lauta in primis ſupellectile, pulcherrimiſque rebus auctum*

fuerit, *iiſque non ad ſpeciem* & *fucum*, *ſed ad veritatem utilitatemque compoſitis*; *Horto nempe* Botanico, *Theatris tribus* Chemico, *Anatomico*, *Chirurgico*; *Muſeis duobus altero Hiſtoriae Naturalis*, Experimentalis Phyſicae *altero*, *gemina Schola*, Clinica *nimirum*, & Obſtetricia *demum Bibliotheca*; *ne quid poſtremo tantorum operum abſolutioni ac perennitati deeſſet*, *hoc ipſo anno inſo-*

litum illi decus & praeſi-dium comparaſti nova re-pente excitata eaque illuſtri & ſplendida Typographia. *Quae omnia tam brevi in-choata & perfecta dum no-ſtrates & exteri novarum re-rum fama huc confluentes attente circumſpiciunt, taci-to quodam ſenſu admonen-tur, non potuiſſe aliunde res tantas, tamque profuſos & pluſquam Regios ſumptus quam ab* Austriaca *Liberali-tate & magnificentia profi-*

ciſci. Harum Tu haeres virtutum, & Majorum Tuorum gloriae ac praeſertim Maternae *cuſtos & aemulator, iſto flore aetatis, iſta vividae indolis laetitia, acerrimo iſto rerum omnium ſenſu admirabilique ſolertia illud jam es aſſequutus, ut in magnis quibuſque inceptis omnes a Te omen capere poſſint. Quae cum ita ſint, haud gravate feres,* Ferdinande Clementissime, *privatum hominem nihilque ſibi tribuen-*

tem tanti PRINCIPIS *patroci-nium ad leves haſce litterulas poſtulare. Sic autem habeto, me mortalium beatiſſimum futurum, ſi infirmos hoſce ſomniculoſi ac timidioris ingenii conatus, publicae lucis diſcrimen magnopere formidantes auſpiciorum Tuorum felicitate fortunaveris. Vale Tibi, populiſque.*

Dabam Papiae ex Regia Univerſitatis Bibliotheca Id. Sept. Ann. MDCCLXXX.



MONITUM.

AD deſignandas poteſtates 2:am, 3:am &c. ſinuum & coſinuum arcus cujuslibet φ uſurpavi promiſcue tum *ſin.*$^2\,\varphi$, *ſin.*$^3\,\varphi$, *cos.*$^2\,\varphi$, *cos.*$^3\,\varphi$ frequentius, tum *ſin.* φ^2, *ſin.* φ^3, *cos.* φ^2,

cos. φ^3 parcius. Praeterea ubicumque occurrit quantitas *ſin.* $2\psi^2$, *ſin.* $2\psi^3$, *cos.* $2\psi^2$, *cos.* $2\psi^3$, intelligi volo poteſtatem ſecundam & tertiam ejus ſinus vel coſinus, qui ad arcum aut angulum 2ψ refertur.

Id monendum duxi, ne ſpecierum inconſtantia, quae mihi ad res quam ad ſigna attentiori obrepſit & impoſuit, perturbationem pareret, & perſpicuitati officiens legentes remoraretur.

Quod ad typorum correctionem attinet, curam hanc moleſtiſſimam mihique fere intolerabilem pro ſua in

me obſervantia voluit alacriter ac ſtrenue ſuſcipere diſcipulus mei amantiſſimus Ferdinandus Meſſia Neapolitanus, ex Olivetana Familia, ſingulari fide homo, ingeniique acumine & judicii vi ſupra aetatem praecellens. Is in ſe recipit depoſito prope pignore, erratum vel omnino nullum, vel certe nullius momenti irrepſiſſe; quod in hujuſmodi libris optare quidem, ſperare vix poſſumus.

Ceterum ſi inter legendum animadvertes, me a Geometrarum maximis alicubi diſſentire, naevumque unum aut alterum in immortalibus il-

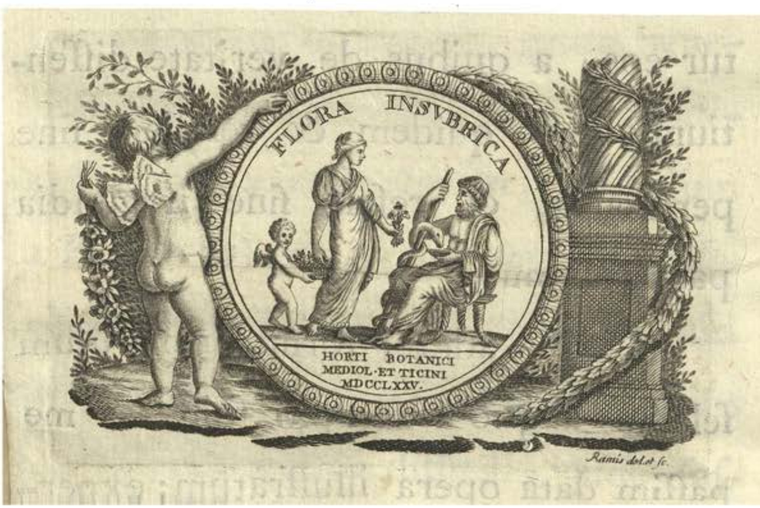
lorum Operibus notare, cave putes, voluiſſe me Viris doctiſſimis & famae ſecuris, a quibus longiſſime abſum, dicam impingere. Sancte enim affirmo, ſine eorum laboribus & vigiliis, unde & haec noſtra manarunt, pagellas haſce nunquam lucem fuiſſe aſpecturas. Quod eo dico, ut omnes intelligant, grata beneficii recordatione nihil mihi eſſe antiquius: *eſt enim benignum*, ut dictum a PLINIO (*a*), *& plenum ingenui pudoris fateri per quos profeceris*. Si quid porro peccatum a nobis fuerit (pluries autem fuiſſe in re difficili, valde ſuſ-

(*a*) In Praef. Hiſt. Nat.

picamur), quo minus eadem lege, id eſt ſine acerbitate & contumelia in nos agatur, non recuſamus; quod ſi non modo ſine convicio, ſed humaniter quoque & amice moneamur, gratias etiam habebimus. Sit iſta, dicam cum Tullio, in Graecorum levitate perverſitas, qui maledictis inſectantur eos, a quibus de veritate diſſentiunt: nos quidem & refellere ſine pertinacia, & refelli ſine iracundia parati ſumus.

Unum addo, formularum uſum ſelecto aliquo exemplo fuiſſe a me paſſim data opera illuſtratum: exper-

tus enim didici, in quaeſtionibus phy-ſico-mathematicis paullo abſtruſioribus tironibus etiam acutioribus, niſi alias fuerint diuturna exercitatione ſubacti, aquam haerere, ubi ab universa rei forma ad peculiare & abſolutum aliquid provocantur.



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PLATO in Philebo

Πασῶν που τεχνῶν ἄν τις ἀριθμητικὴν χωρίζῃ, καὶ μετρητικὴν, καὶ στατικὴν, ὡς ἔπος εἰπεῖν, φαῦλον τὸ καταλειπόμενον ἑκάστης ἂν γίγνοιτο.

Si quis ab omnibus Artibus ſegregaret *numerandi*, *dimetiendique*, & *ponderandi* peritiam, vile quiddam eſſet quod uniuſcujuſque reſtaret.

Ex verſ. MARSILII FICINI.

DISQUISITIO I.

DE CALORIS DIURNI SOLARIS IN VARIIS TERRAE LOCIS AESTIMATIONE ET COMPARATIONE.

1. PRofundi vir ingenii ſubactique judicii Edmundus HALLEJUS, quo nemo propius inter Britannos Philoſophos ad Magni NEWTONI gloriam acceſſiſſe videtur, anno 1693. omnium primus ſubtiliſſimam ſuſcepit inveſtigationem *de caloris ſolaris magnitudine in omnibus latitudinibus terreſtribus proportionali*, & Tranſactionum Philoſophicarum num. 203. elegantem protulit ſyntheticam Problematis ſpecioſi ac difficilis conſtructionem (*a*). Poſt HALLEJI fata Geometra nobilis Thomas SIMPSONIUS in eximio Opere *De Fluxionum Doctrina* Problema retractavit analytice, formulamque exhibuit ſimpliciſſimam ex-

(a) Phil. Trans. num. 203. *The proportional Heat of the Sun in all Latitudes*.

primentem Solaris aeſtus vim in variis Terrae locis pro dato quolibet anni die, vel parte diei (*b*). Sed cum in eo eſſet SIMPSONIUS, ut fluxionionalis formulae momentaneum calorem exhibentis fluentem magnitudinem definiret, omiſſa conſtanti, quae non erat prorſus negligenda, humani aliquid paſſus eſt. Statui itaque, Problema iterum via analytica aggredi, Tabulamque praeterea ſupputare, quae rationem exhibeat inter calorem diurnum utrumque ſub Æquatore, & Polo pro declinationibus ſingulis ſolaribus ab 1° uſque ad 23.° 28′.

2. Jam primum omnium ſatis conſtat, intendi calorem Solis creſcente ipſius ſupra horizontem altitudine; & in ea ſane ratione intendi, qua augetur ſinus altitudinis illius: neque enim audiendus eſt vir caetera doctus Fatio de DUILLIER, qui in Tractatu *De Murorum ad fovendas Arbores fructiferas inclinatione* (*c*), ſimplici rationi ſinus elevationis Solaris rationem duplicatam ſubſtituit, contra quem egregie diſputat, quaeſtionemque omnino dirimit vir acer & ingenioſus MAIRANUS in tertio *De generali Caloris Solaris Cauſſa* Commentario (*d*). Con-

(b) SIMPSON *The Doctrine and Application of Fluxions* §. 470.

(c) *Fruit Walls improved by inclining them to the Horizon* p. 39.

(d) *Mém. de l' Accad. des Sciences de Paris* ann. 1765.

ſtat praeterea, majorem rurſus fieri inſtantaneum Solis calorem, quo longior eſt momenti, vel tempuſculi infiniteſimi duratio, ut nimirum augeatur in ratione directa tempuſculi. Hinc vero conſequens eſt, momentaneum Solis aeſtum rationem ſequi compoſitam ex ſinu altitudinis & ex tempuſculo conjunctim, riteque repraeſentari per factum ex ſinu altitudinis in tempuſculum.

3. Supereſſet ſane in hac ſupputatione elementum alterum expendendum, debilitatio ſcilicet radiorum ſolarium aerem permeantium, qui eo diſperguntur magis atque intercipiuntur, quo minor eſt Solis elevatio. Sed praeter quam quod aliqua hic habetur compenſatio, quum ex majori radiorum interceptorum copia terra quidem minus, aer autem magis incaleſcat, adeo perplexum incertumque deprehenditur hujuſmodi elementum, ut nulli certae legi aut rationi ſubjici queat. Liquet id ex collatione Tabularum Bouguerianae, & Lambertianae (e) quae varios exhibent debilitationis gradus radiorum ſolarium varia obliquitate athmoſphaeram tranantium, quarum tabularum maxima eſt & vix credibilis diſcrepantia.

(e) BOUGUER *Traité d' Optique ſur la Gradation de la Lumière* Liv. III. Sect. IV.

LAMBERT *Photometria* §. 886.

4. Reſtat ergo, ut ardorem Solis inſtantaneum proportionalem ſtatuamus facto quod oritur ex multiplicatione tempuſculi per ſinum altitudinis Solaris, hocque productum differentiale ad integrationem rite perducamus: ita enim formulam nanciſcemur, quae Solis calorem repraeſentabit intervallo quolibet diei perdurantem, indeque licebit comparationem Caloris diurni Æquatorii & Polaris perſpicue deducere.

Fig. 1. 5. Sit itaque in triangulo ſphaerico SPZ S Sol, P Polus, Z Zenit, itaut latus PZ ſit complementum latitudinis, latus PS complementum declinationis, latus ZS complementum elevationis Solis ſupra horizontem, vocataque loci latitudine λ, ſolis declinatione δ, & elevatione ε orietur

ſin. PZ = cos. lat. = cos. λ
ſin. PS = cos. decl. = cos. δ
cos. PZ = ſin. lat. = ſin. λ
cos. PS = ſin. decl. = ſin. δ
cos. ZS = ſin. elev. = ſin. ε

Jam vero notum eſt ex Sphaericorum Doctrina, in triangulo quolibet SPZ ſequentem ſe prodere aequalitatem cos. ZS = ſin. PZ ſin. PS cos. P + cos. PZ cos. PS, hoc eſt, facto angulo horario P, vel SPZ = z,

ſin. ε = cos. λ cos. δ cos. z + ſin. λ ſin. δ

6. Hoc poſito quum calor Solis inſtantaneus ſit uti factum ex ſinu elevationis in tempuſculum, ducto ſin. ε in elementum anguli horarii prodibit differentialis formula $-$ dz ſin. ε, hoc eſt $-$ dz cos. z cos. λ cos. δ $-$ dz. ſin. λ ſin. δ, in qua differentiale dz ſumitur negative, quod creſcente Solis altitudine decreſcit horarius angulus. Hujus porro formulae integrale facile invenitur $= -$ cos. λ cos. δ ſin. z $-$ z ſin. λ ſin. δ $+$ conſt.; talis igitur erit expreſſio caloris perſeverantis ab ortu Solis uſque ad elevationem ε. Quum autem in ipſo ortus momento calor ſit nullus, tuncque horarius angulus z abeat in arcum datum ſemidiurnum quem nominabimus h, fit iccirco conſt. $=$ cos. λ cos. δ ſin. h $+$ h ſin. λ ſin. δ. Quapropter Solis calor ab ortu uſque ad elevationem ε repraeſentatur eleganti formula

(ſin. h $-$ ſin. z) cos. λ cos. δ $+$ (h $-$ z) ſin. λ ſin. δ.

7. Si in hac formula accipitur horarius angulus z $=$ o, quod fit ubi Sol loci meridianum attingit, ea convertitur in alteram ſimpliciorem ſin. h cos. λ cos. δ $+$ h ſin. λ ſin. δ exprimentem Solaris aeſtus quantitatem intervallo dimidiae diei aggregatam & collectam. Et quoniam mutata Solis declinatione e boreali in auſtralem ſin. δ negativus evadit, hinc eſt, ut Solis calor intervallo integrae diei perſeverans

generalius, commodiuſque exprimi queat per formulam $2\ \text{ſin.}\ h \cos.\ \lambda \cos.\ \delta \pm 2\ h\ \text{ſin.}\ \lambda\ \text{ſin.}\ \delta$.

8. Ex hiſce porro conſequitur, diurnum Solis calorem in loco quovis terreſtri pro data qualibet Solis declinatione facillime comparari poſſe cum calore diurno loci alius cujuſcumque pro data alia quavis declinatione, ita ut, ſi h', λ', δ' deſignent angulum alium horarium, aliamque latitudinem, & declinationem, calor primus ſe habeat ad alterum quemadmodum $2\ \text{ſin.}\ h \cos.\ \lambda \cos.\ \delta \pm 2\ h\ \text{ſin.}\ \lambda\ \text{ſin.}\ \delta$ ſe habet ad $2\ \text{ſin.}\ h' \cos.\ \lambda' \cos.\ \delta' \pm 2\ h'\ \text{ſin.}\ \lambda'\ \text{ſin.}\ \delta'$. Igitur dictis C, C' diurnis caloribus haec ſemper ſtabit analogia $C : C' :: \text{ſin.}\ h \cos.\ \lambda \cos.\ \delta \pm h\ \text{ſin.}\ \lambda\ \text{ſin.}\ \delta : \text{ſin.}\ h' \cos.\ \lambda' \cos.\ \delta' \pm h'\ \text{ſin.}\ \lambda'\ \text{ſin.}\ \delta'$, in qua binorum ſignorum $\pm$ affirmativum uſurpatur, ſi declinatio ſit ejuſdem nominis cum latitudine, borealis cum boreali, auſtralis cum auſtrali; & negativum accipitur, ſi contraria.

9. Quum autem pro variis Solis declinationibus variet ipſius a Tellure diſtantia, hujuſque diſtantiae quadratis reciproce reſpondeat Solaris caloris proportio, quemadmodum Phyſicis perſpectum eſt; icciroo aſſumptis D, D' pro diſtantiis Solis a Terra, accuratiorem nanciſcimur analogiam

$$C : C' : : \frac{\text{ſin. h cos. } \lambda \text{ cos. } \delta \pm \text{h ſin. } \lambda \text{ ſin. } \delta}{D^2} . \frac{\text{ſin. h' cos. } \lambda' \text{ cos. } \delta' \pm \text{h' ſin. } \lambda' \text{ ſin. } \delta'}{D'^2},$$

quae caloris Solaris diurni comparationem in binis locis quibuſcumque dato quolibet anni die patefaciet.

10. Quaeratur ex. gr. proportio caloris Solaris die ſolſtitiali aeſtivo in Urbibus Petropoli, & Ticino. Ante omnia inveniendus eſt arcus ſemidiurnus h, h' datae loci latitudini, dataeque Solis declinationi reſpondens. Ad hoc aſſequendum, ex formula §. 5. elicio $\text{cos. } z = \frac{\text{ſin. } \varepsilon}{\text{cos. } \lambda \text{ cos. } \delta} - \frac{\text{ſin. } \lambda \text{ ſin. } \delta}{\text{cos. } \lambda \text{ cos. } \delta} = \frac{\text{ſin. } \varepsilon}{\text{cos. } \lambda \text{ cos. } \delta}$ $- \text{tang. } \lambda \text{ tang. } \delta$: ſi jam nulla eſſet horizontalis Solis parallaxis, nullaque horizontalis refractio, poſita centri Solaris altitudine $\varepsilon = 0$, prodiret $z = h$, & $\text{cos. h} = - \text{tang. } \lambda \text{ tang. } \delta$. Verum quoniam horizontalis parallaxis centrum Solis deprimit minutis ſecundis circiter $8\frac{1}{2}$, & refractio illud attollit minutis primis 33, eapropter neglecta parallaxi, quae prae refractione contemni tuto poteſt, factaque altitudine $\varepsilon = - 33'$, orietur arcus ſemidiurni $\text{cos. h} = - \frac{\text{ſin. } 33'}{\text{cos. } \lambda \text{ cos. } \delta}$ $- \text{tang. } \lambda \text{ tang. } \delta$. Eſt igitur Ticini

λ . . . . 45.° 11'

δ	23.° 28′
log. ſin. 33′	7, 9822334
log. cos. λ	9, 8480909
log. cos. δ	9, 9625076
log. cos. λ cos. δ	9, 8105985
log. $\frac{\text{ſin. } 33'}{\cos.\lambda \cos.\delta}$	8, 1716349
$-\frac{\text{ſin. } 33'}{\cos.\lambda \cos.\delta}$	—0, 014847
log. tang. λ	10, 0027793
log. tang. δ	9, 6376106
log. tang. λ tang. δ	9, 6403899
— tang. λ tang. δ	—0, 43691

Quamobrem ſit cos. h = − 0, 014847 − 0, 43691 = − 0, 451757, ac denique h = 116.° 51′.

Petropoli vero quum ſit λ' = 59.° 56′, invenitur

log. ſin. 33′	7, 9822334
log. cos. λ'	9, 6998441
log. cos. δ	9, 9625076
log. cos. λ' cos. δ	9, 6623517

$\log.\ \frac{\text{ſin.}\ 33'}{\cos.\lambda'\cos.\delta}$ 8, 3198817

$-\ \frac{\text{ſin.}\ 33'}{\cos.\lambda'\cos.\delta}$ . . . . —0, 020887

log. tang. λ' 10, 2373944

log. tang. δ 9, 6376106

log. tang. λ' tang. δ 9, 8750050

— tang. λ' tang. δ . . . . —0, 74990

Quare cos. $h' = -0,020887 - 0,7499 = -0,770787$, & conſequenter $h' = 140°.\ 25'$.

Quum vero aſſumpto circuli radio = 1, ſemiperipheria ſit 3, 14159, convertitur arcus ſemidiurnus h in partes radii ope analogiae 180°. : 116°. 51' : : 10800' : 7011' : : 3, 14159 : h = 2, 0394; ſicque per alteram analogiam 180°. : 140°. 25' : : 10800' : 8425' : : 3, 14159 : $h' = 2,\ 4507$. Inventis porro arcubus ſemidiurnis h, h', quoniam in hac hypotheſi eſt $D = D'$, & $\delta = \delta'$, oritur ideo C : C' : : ſin. h cos. λ cos. δ + h ſin. λ ſin. δ : ſin. h' cos. λ' cos. δ + h' ſin. λ' ſin. δ, ſeu dividendo per cos. δ, fit C : C' : :

ſin.h cos.λ + h ſin.λ tang.δ : ſin.h' cos.λ' + h' ſin.λ' tang.δ

Erit itaque.

log. ſin. h 9, 9504583

log. cos. λ 9, 8480909

log. ſin. h cos. λ	9, 7985492
ſin. h. cos. λ	0, 62885
log. h	0, 3095024
log. ſin. λ	9, 8508702
log. tang. δ	9, 6376106
log. h ſin. λ tang. δ	9, 7979832
h ſin. λ tang. δ	0, 62803
ſin. h cos. λ + h ſin. λ tang. δ . .	1, 25688

log. ſin. h'	9, 8042757
log. cos. λ'	9, 6998441
log. ſin. h' cos. λ'	9, 5041198
ſin. h' cos. λ'	0, 31924
log. h'	0, 3892902
log. ſin. λ'	9, 9372385
log. tang. δ	9, 6376106
log. h' ſin. λ' tang. δ	9, 9641393
h' ſin. λ' tang. δ	0, 92074
ſin. h' cos. λ' + h' ſin. λ' tang. δ	1, 23998

Quamobrem oritur C : C' : : 1, 25688 : 1, 23998,

hoc eſt Ticini calor diei ſolſtitialis aeſtivi ſe habet ad calorem ejuſdem diei Petropoli, quemadmodum ſere 126 ad 124: quod ſane mirum eſſe non debet, quum in hac hypotheſi ejus dumtaxat caloris habeatur ratio, qui die unico ſolſtitiali cumulatur, ſecluſo penitus calore diebus omnibus praecedentibus collecto.

11. Quaeratur ſecundo loco in eadem Urbe Ticino proportio caloris diei aeſtivi ſolſtitialis, & diei ſolſtitialis hyberni ſeu ratio C : C'. In hac hypotheſi habetur

$$C : C' :: \frac{\text{ſin. } h \cos. \lambda \cos. \delta + h \text{ ſin. } \lambda \text{ ſin. } \delta}{D^2}$$

$$: \frac{\text{ſin. } h' \cos. \lambda \cos. \delta - h' \text{ ſin. } \lambda \text{ ſin } \delta}{D'^2}, \text{ videlicet}$$

$$C : C' :: \frac{\text{ſin. } h + h \text{ tang. } \lambda \text{ tang. } \delta}{D^2} : \frac{\text{ſin. } h' - h' \text{ tang. } \lambda \text{ tang. } \delta}{D'^2}.$$

Conſtat autem ex Aſtronomicis Tabulis, logarithmum diſtantiae Solis a Terra aeſtivi ſolſtitii die, hoc eſt log. D aequari 0, 007161, & logarithmum ejus diſtantiae ſolſtitii hyemalis die, nimirum log. D' aequalem eſſe 9, 992716, aſſumpta diſtantia mediocri = 1. Hinc igitur conſequimur

ſin. h	0, 8921920
log. h	0, 3095024
log. tang. λ tang. δ	9, 6403899

log. h tang. λ tang. δ	9, 9498923
h tang. λ tang. δ	0, 8910300
ſin. h + h tang. λ tang. δ	1, 7832220
log. (ſin. h + h tang. λ tang. δ) . .	0, 2512001
log. D^2	0, 0143220
log. $\frac{\text{ſin. h + h tang. }\lambda\text{ tang. }\delta}{D^2}$	0, 2368781
$\frac{\text{ſin. h + h tang. }\lambda\text{ tang. }\delta}{D^2}$	1, 7253

Poſtremus analogiae terminus ſupputabitur invento prius h\`, ſeu arcu ſemidiurno hyemalis ſolſtitii tempore ; habetur autem cos. h\` $= - \frac{\text{ſin. } 33'}{\text{cos. }\lambda\text{ cos. }\delta}$ + tang. λ tang. δ = − 0, 014847 + 0, 43691 = 0, 422063 ; atque inde h\` = 65.° 2\`, qui in partes radii converſus evadit = 1, 1350. Jam igitur nanciſci licet

ſin. h\`	0, 9065535
log. h\`	0, 0549959
log. tang. λ tang. δ	9, 6403899
log. h\` tang. λ tang. δ	9, 6953858
— h\` tang. λ tang. δ	−0, 49589

ſin. h' − h' tang. λ tang. δ	0, 4106635
log. (ſin. h' − h' tang. λ tang. δ) . .	9, 6134860
log. D'²	9, 9854320
log. $\frac{\text{ſin. h'} - \text{h' tang. } \lambda \text{ tang. } \delta}{D'^2}$	9, 6280540
$\frac{\text{ſin. h'} - \text{h' tang. } \lambda \text{ tang. } \delta}{D'^2}$	0, 42467

Propterea fit C : C' : : 1, 7253 : 0, 42467, nimirum aeſtivus diei ſolſtitialis calor Ticini invenitur plus quam quadruplo major calore hyberno diei ſolſtitialis oppoſiti.

12 Reliquum eſt, ut tabulam ſupputemus exhibentem comparationem Caloris diurni Æquatorii, & Polaris pro ſingulis Solis declinationibus ab 1° ad 23.° 28'. Ad hoc praeſtandum animadverto, caloris expreſſionem ſin. h cos. λ cos. δ + h ſin. λ ſin. δ ſub Æquatore mutari in ſin. h cos. δ ob λ = 0, alteramque expreſſionem caloris ſub Polo, ſin. h' cos. λ' cos. δ + h' ſin. λ' ſin. δ abire in h' ſin. δ ob λ' = 90.°. Hinc itaque ſe prodit proportio C : C' : : ſin. h cos. δ : h' ſin. δ : : ſin. h : h' tang. δ. Perſpicuum porro eſt, ſub Polo fieri h' = 180°, ſeu in partibus radii, h' = 3, 14159, adeoque C : C' : : ſin. h : 3, 14159 tang. δ.

Invenitur autem h pro ſingulis Solis declinationibus ex formula (§ 10) $\cos. h = -\frac{\text{ſin. } 33'}{\cos. \lambda \cos. \delta} - \text{tang.} \lambda \text{ tang.} \delta$, quae ob $\lambda = 0$ degenerat in $\cos. h = -\frac{\text{ſin. } 33'}{\cos. \delta}$. Ex hiſce facile conſtruitur ſequens

Tabula Comparativa

Caloris diurni Aequatorii, & Polaris pro ſingulis Solis declinationibus.

Declinatio Solis ſeu δ	*Arcus ſemi-diurnus ſub Aequatore, ſeu* h	*Calor diurnus Aequatorius ſeu* ſin. h	*Calor diurnus Polaris ſeu* 3,14159 tang. δ
1° . .	90° 33'	0,999954 . .	0,054837
2° . .	90° 33'	0,999954 . .	0,10971
3° . .	90° 33'	0,999954 . .	0,16464
4° . .	90° 33'	0,999954 . .	0,21968
5° . .	90° 33'	0,999954 . .	0,27485

Declinatio Solis ſeu δ	*Arcus ſemi-diurnus ſub Aequatore, ſeu* h	*Calor diurnus Aequatorius ſeu* ſin. h	*Calor diurnus Polaris ſeu* 3,14159 tang.
6° . .	90°. 33'	0,999954 . .	0,33020
7° . .	90°. 33'	0,999954 . .	0,38574
8° . .	90°. 33'	0,999954 . .	0,44152
9° . .	90°. 33'	0,999954 . .	0,49758
10° . .	90°. 34'	0,999951 . .	0,55395
11° . .	90°. 34'	0,999951 . .	0,61067
12° . .	90°. 34'	0,999951 . .	0,66777
13° . .	90°. 34'	0,999951 . .	0,72530
14° . .	90°. 34'	0,999951 . .	0,78329
15° . .	90°. 34'	0,999951 . .	0,84179
16° . .	90°. 34'	0,999951 . .	0,90084
17° . .	90°. 35'	0,999948 . .	0,96048
18° . .	90°. 35'	0,999948 . .	1,02077
19° . .	90°. 35'	0,999948 . .	1,08174
20° . .	90°. 35'	0,999948 . .	1,14345
21° . .	90°. 35'	0,999948 . .	1,20594

Declinatio Solis ſeu δ	*Arcus ſemi-diurnus ſub Aequatore, ſeu* h	*Calor diurnus Aequatorius ſeu* ſin. h	*Calor diurnus Polaris ſeu,* 3,14159 tang. δ
22° . .	90°. 36′	0,999945 . . .	1,26929
23° . .	90°. 36′	0,999945 . .	1,33353
23°. 28′	90°. 36′	0,999945 . .	1,36414



DIS-

DISQUISITIO II.

DE CALORE ANNUO SOLARI.

1. QUum in praecedenti Diſquiſitione Caloris *Diurni* vim ac menſuram in variis Terrae locis rite ſupputaverimus, ejuſdemque comparationem ſub Æquatore & Polo pro ſingulis Solis declinationibus luculenter deduxerimus, reliquum nunc eſt, ut *Annuum* quoque Calorem in utroque loco expendamus, ejuſque intenſionem accurata quoad fieri poteſt analyſi dimetiamur.

2. Ut autem in hac paullo abſtruſiori indagine ordinatim ac tuto progrediamur, neceſſe eſt in anteceſſum conſtituere, radiorum ſolarium numerum tempore quovis finito in Terram illabentium rationem directam ſequi anguli illius, quem interea temporis Terra circa Solem deſcribit: quod quidem elegans Theorema ita breviter demonſtratur: Accipiatur in Orbita Terreſtri, ſive Ecliptica *APBF* arcus infiniteſimus *Pp*, ductiſque ab umbilico, ſeu centro Solis *S* radiis vectoribus *SP*, *Sp* deſcribatur minimus circuli arcus *Pg*, qui trianguli infiniteſimi *PSp* altitudinem

repraeſentabit accepto latere Sp pro baſi; inde vero triangulum ipſum invenitur aequale facto $\frac{1}{2} Sp . Pg$, vel $\frac{1}{2} SP . Pg$. Porro ex Kepleriana Lege areola PSp, hoc eſt factum $\frac{1}{2}$ $SP . Pg$ exprimit tempus, quo Tellus minimum Eclipticae arcum Pp deſcribit: ac praeterea perſpicuum eſt, radiorum ſolarium numerum momento temporis in Terram illabentium eſſe ut tempuſculum directe, & diſtantiae quadratum inverse. Igitur radiorum numerus in Terram irruentium eo tempuſculo, quo Tellus movetur per minimum arcum Pp, erit uti $\frac{\frac{1}{2} SP.Pg}{SP^2}$, hoc eſt uti $\frac{Pg}{SP}$. Eſt autem circuli arcus Pg ut ſemidiameter SP, & angulus PSg conjunctim; conſequenter numerus ille radiorum fit uti $\frac{PSg.SP}{SP}$, nimirum uti angulus PSg. Quum id ſingulis momentis contingat, liquet, numerum radiorum, qui tempore quolibet finito in Terram emittuntur, dum ex. gr. Tellus Eclipticae arcum AP percurrit, fore uti angulus ASP interea deſcriptus.

3. Hoc in anteceſſum poſito, cogitemus nunc triangulum ſphaericum rectangulum ex tribus arcubus Circulorum Sphaerae Maximorum, Eclipticae, Æquatoris, & Declinationis; cujus quidem trianguli hypothenuſa eſt arcus Eclipticae ab initio Arietis uſque

ad Solis locum, nimirum longitudo Solis, latus unum eſt arcus Circuli Declinationis a Solis loco uſque ad Æquatorem, videlicet declinatio Solis, ac latus alterum eſt arcus Æquatoris ab initio Arietis uſque ad Circulum Declinationis. Vocetur jam longitudo Solis v, Eclipticae obliquitas φ, & Solis declinatio δ; & quum in ſphaerico quovis triangulo ſinus laterum ſint inter ſe uti ſinus angulorum lateribus oppoſitorum, fiet iccirco

$$1 : \mathit{ſin}.\varphi :: \mathit{ſin}.v : \mathit{ſin}.\delta .$$

Hinc itaque colligitur $\mathit{ſin}.\delta = \mathit{ſin}.\varphi\ \mathit{ſin}.v$, & $\mathit{cos}.\delta = \sqrt{(1 - \mathit{ſin}.^2\varphi\ \mathit{ſin}.^2 v)}$. Quum vero Calor diurnus aequatorius in ſuperiori Diſquiſitione inventus fuerit $= \mathit{ſin}.h\ \mathit{cos}.\delta$, & Calor diurnus Polaris $= 3,14159\, \mathit{ſin}.\delta = \pi\, \mathit{ſin}.\delta$, ſumpto π pro circuli ſemiperipheria; eapropter ſubſtitutis valoribus modo erutis $\mathit{ſin}.\delta$, & $\mathit{cos}.\delta$, orietur Calor Æquatorius diurnus $= \mathit{ſin}.h \sqrt{(1 - \mathit{ſin}.^2\varphi\ \mathit{ſin}.^2 v)}$, & Calor diurnus Polaris $= \pi\, \mathit{ſin}.\varphi\ \mathit{ſin}.v$.

4. Jam vero quum Sol ſingulis diebus variam longitudinem aſſequatur, & ejus longitudinis variatio intra diem ſat parva ſit, uti ex Aſtronomicis conſtat, licebit variationem eandem inſtar differentialis tractare, & aequalem ponere longitudinis elemento dv. Porro quoniam huic diurnae variationi dv (§. 2.)

proportionalis eſt radiorum Solarium numerus intra diei ſpatium in Terram vibratorum, perſpicuum eſt, ſi calor diurnus Æquatoris, & Poli ducatur in dv, gigni elementum caloris annui indefiniti, qui nimirum reſpondet tempori, quo Sol ad indeterminatam longitudinem v pervenit. Quapropter elementum caloris annui ſub Æquatore erit $dv\, ſin.h \sqrt{(1 - ſin.^2 \varphi\, ſin.^2 v)}$, & ſub Polis erit $\pi dv\, ſin.\varphi\, ſin.v$. Harum formularum integratio ita erit abſolvenda, ut integralia evaneſcant una cum ipſa longitudine v; fingimus enim, actionem Solis calefacientis tunc incipere cum Sol Arietis ſignum ingreditur, ſive cum nulla eſt ejuſdem longitudo. Caeterum vel me non monente patet, nullam in hoc calculo rationem haberi caloris, qui noctis tempore a corporibus decedit, nullam actionis, qua calor calori additus hujus vim intendit, nullam denique caloris reflexi, qui directum diverſimode afficit, augetque vehementer; quae ſane omnia in calculum introducta Problema efficerent inextricabile, implexum, & hodiernae Analyſi penitus inacceſſum.

5. Quum igitur Calor Æquatorius toto illo tempore indefinito, quo Sol deſcribit Eclipticae arcum v exprimatur integrali formula $\int dv\, ſin.h \sqrt{(1 - ſin.^2 \varphi\, ſin.^2 v)}$; ad hujus formulae integrationem rite abſolvendam animadverto eſſe $\sqrt{(1 - ſin.^2 \varphi\, ſin.^2 v)} = 1 - \frac{ſin.^2 \varphi\, ſin.^2 v}{2}$

$$-\frac{ſin.^4\varphi\, ſin.^4 v}{2.2.2.}-\frac{ſin.^6\varphi\, ſin.^6 v}{2.2.2.2.}-\frac{5\, ſin.^8\varphi\, ſin.^8 v}{2.2.2.2.2.4}$$

$$-\frac{7\, ſin.^{10}\varphi\, ſin.^{10} v}{2.2.2.2.2.2.4}-\frac{7.9 ſin.^{12}\varphi\, ſin.^{12} v}{2.2.2.2.2.2.2.4.6}$$

$$-\frac{9.11 ſin.^{14}\varphi\, ſin.^{14} v}{2.2.2.2.2.2.2.2.4.6}-\frac{9.11.13. ſin.^{16}\varphi\, ſin.^{16} v}{2.2.2.2.2.2.2.2.2.4.6.8}$$

$$-\frac{11.13.15 ſin.^{18}\varphi\, ſin.^{18} v}{2.2.2.2.2.2.2.2.2.2.4.6.8}-\&c.$$

Tum duco terminos hoſce omnes in $dv\, ſin.h$, & ſingulorum productorum integrationem aggredior.

6. Itaque conſtat ex Integrali Calculo, eſſe

I.

$$\int dv\, ſin.h = v\, ſin.h$$

II.

$$\int dv\, ſin.^2 v = -\tfrac{1}{2}\, ſin.v\, cos.v + \tfrac{1}{2} v;$$ atque iccirco

$$-\tfrac{1}{2} ſin.h\, ſin.^2\varphi \int dv\, ſin.^2 v = \frac{ſin.h\, ſin.^2\varphi\, ſin.v\, cos.v}{2.2}$$

$$-\frac{v\, ſin.h\, ſin.^2\varphi}{2.2}$$

III.

$$\int dv\, ſin.^4 v = -\tfrac{1}{4}\, ſin.^3 v\, cos.v - \frac{3 ſin.v\, cos.v}{2.4} + \frac{3v}{2.4};$$

$$-\frac{ſin.h\, ſin.^4\varphi}{2.2.2}\int dv\, ſin.^4 v = \frac{ſin.h\, ſin.^4\varphi\, ſin.^3 v\, cos.v}{2.2.2.4}$$

$$+ \frac{3\, ſin.h\, ſin.^4 \varphi\, ſin.v\, cos.v}{2.2.2.2.4} - \frac{3\, v ſin.h\, ſin.^4 \varphi}{2.2.4.4.}$$

IV.

$$\int dv\, ſin.^6 \varphi = -\tfrac{1}{6} cos.v\, ſin.^5 v - \frac{5\, cos.v\, ſin.^3 v}{4.6}$$

$$- \frac{5.3\, cos.v\, ſin.v}{2.4.6} + \frac{5.3 v}{2.4.6.};$$

$$- \frac{ſin.h\, ſin.^6 \varphi}{2.2.2.2} \int dv\, ſin.^6 v = \frac{ſin.h\, ſin.^6 \varphi\, cos.v\, ſin.^5 v}{2.2.2.2.6}$$

$$+ \frac{5\, ſin.h\, ſin.^6 \varphi\, cos.v\, ſin.^3 v}{2.2.2.2.4.6} + \frac{5.3\, ſin.h.\, ſin.^6 \varphi\, cos.v\, ſin.v}{2.2.2.2.2.4.6}$$

$$- \frac{5.3 v\, ſin.h\, ſin.^6 \varphi}{2.2.2.2.2.4.6}$$

V.

$$\int dv\, ſin.^8 v = -\tfrac{1}{8} cos.v\, ſin.^7 v - \frac{7 cos.v\, ſin.^5 v}{6.8}$$

$$- \frac{7.5 cos.v\, ſin.^3 v}{8.6.4} - \frac{7.5.3\, cos.v ſin.v}{8.6.4.2} + \frac{7.5.3 v}{8.6.4.2};$$

$$- \frac{5 ſin.h\, ſin.^8 \varphi}{2.2.2.2.2.4} \int dv\, ſin.^8 v = \frac{5 ſin.h\, ſin.^8 \varphi\, cos.v\, ſin.^7 v}{2.2.2.2.2.4.8}$$

$$+ \frac{5.7 ſin.h\, ſin.^8 \varphi\, cos.v\, ſin.^5 v}{2.2.2.2.2.4.6.8} + \frac{5.5.7\, ſin.h\, ſin.^8 \varphi\, cos.v\, ſin.^3 v}{2.2.2.2.2.4.4.6.8}$$

$$+ \frac{3.5.5.7 ſin.h\, ſin.^8 \varphi\, cos.v\, ſin.v}{2.2.2.2.2.2.4.4.6.8} - \frac{3.5.5.7 v ſin.h\, ſin.^8 \varphi}{2.2.2.2.2.2.4.4.6.8}$$

VI.

&c.

Accipiatur jam horum terminorum ſumma, & vocetur A coefficiens producti $\cos.v\ \text{ſin}.v$, B coefficiens producti $\cos.v\ \text{ſin}.^3v$, C coefficiens producti $\cos.v\ \text{ſin}.^5v$, D coefficiens producti $\cos.v\ \text{ſin}.^7v$, atque ita porro:

hinc habetur $\int dv\ \text{ſin}.h \sqrt{(1 - \text{ſin}.^2\varphi\ \text{ſin}.^2v)}$

$$= A\cos.v\ \text{ſin}.v + B\cos.v\ \text{ſin}.^3v + C\cos.v\ \text{ſin}.^5v + D\cos.v\ \text{ſin}.^7v + \&c.$$

$$\left.\begin{array}{r} + \text{ſin}.h \\ - \dfrac{\text{ſin}.h\ \text{ſin}.^2\varphi}{2.2} \\ - \dfrac{3\,\text{ſin}.h\ \text{ſin}.^4\varphi}{2.2.4.4} \\ - \dfrac{3.3.5\,\text{ſin}.h\ \text{ſin}.^6\varphi}{2.2.4.4.6.6} \\ - \dfrac{3.3.5.5.7\,\text{ſin}.h\ \text{ſin}.^8\varphi}{2.2.4.4.6.6.8.8} \\ - \&c. \end{array}\right\} v.$$

In hac integratione conſtans nulla addenda eſt, quod evaneſcente v Calor omnis una cum integrali indefinito evaneſcit.

7. Quum porro integrale modo inventum exhibeat Calorem Solis Æquatorium toto illo tempore indeterminato, quo Sol ab initio Arietis indefinitam longitudinem v adipiſcitur, facile jam erit Calorem invenire tempori cuivis determinato reſpondentem. Quaeratur ex. gr. Calor ſub Æquatore per totum tempus Anni

dimidii: In hac hypotheſi fit $v = 180° = \pi$, & in praecedenti formula ob $ſin.\ v = 0$, evaneſcunt termini omnes praeter ultimum, qui convertitur in $\pi\ ſin.\ h \left(1 - \frac{ſin.^2 \varphi}{2.2} - \frac{3 ſin.^4 \varphi}{2.2.4.4} - \frac{3.3.5 ſin.^6 \varphi}{2.2.4.4.6.6} - \frac{3.3.5.5.7 ſin.^8 \varphi}{2.2.4.4.6.6.8.8} - \&c. \right) =$ Calori Æquatorio per dimidium annum; proindeque Calor ſub Æquatore per annum integrum repraeſentatur a duplo ipſius formulae, nimirum $2\pi\ ſin.\ h \left(1 - \frac{ſin.^2 \varphi}{2.2} - \frac{3 ſin.^4 \varphi}{2.2.4.4} - \frac{3.3.5 ſin.^6 \varphi}{2.2.4.4.6.6} - \frac{3.3.5.5.7 ſin.^8 \varphi}{2.2.4.4.6.6.8.8} - \&c. \right)$, quae ſane quantitas ſeriem complectitur valde convergentem, ut per ſe liquet.

8. Praeterea inventus eſt (§. 4.) Calor Polaris $= \int \pi\ dv\ ſin.\ \varphi\ ſin.\ v$, nimirum, facta integratione, $= -\pi\ ſin.\ \varphi\ cos.\ v +$ Conſt. Oritur autem Conſt. $= \pi\ ſin.\ \varphi$, quod poſito $v = 0$, etiam Calor fit nihilo aequalis. Quamobrem invenitur Polaris Calor $= \pi\ ſin.\ \varphi\ (1 - cos.\ v)$. Si jam ſumitur $v = 180°$, prodit $cos.\ v = -1$, & $-cos.\ v = 1$; atque ideo quantitas $\pi ſin.\ \varphi\ (1 - cos.\ v)$ mutatur in $2\pi ſin.\ \varphi$. Eapropter Polaris Calor ſex integris menſibus perſeverans repraeſentatur quantitate $2\pi ſin.\ \varphi$, quae ſane

quantitas Calorem quoque totius anni indigitabit, quum altero anni dimidio Sol ſit Polo inconſpicuus, & infra horizontem deliteſcat.

9. Hinc itaque habetur comparatio Caloris annui ſub Æquatore, & Polo; eſt enim annuus Æquatoris Calor ad annuum Poli Calorem quemadmodum eſt

$$2\pi\, \text{ſin.}\, h \left(1 - \frac{\text{ſin.}^2 \varphi}{2.2} - \frac{3\, \text{ſin.}^4 \varphi}{2.2.4.4} - \frac{3.3.5\, \text{ſin.}^6 \varphi}{2.2.4.4.6.6} - \frac{3.3.5.5.7\, \text{ſin.}^8 \varphi}{2.2.4.4.6.6.8.8} - \&c.\right)$$

ad $2\pi\, \text{ſin.}\, \varphi$, ſive uti $\text{ſin.}\, h$

$$\left(1 - \frac{\text{ſin.}^2 \varphi}{2.2} - \frac{3\, \text{ſin.}^4 \varphi}{2.2.4.4} - \&c.\right)$$

ad $\text{ſin.}\, \varphi$, hoc eſt uti $P\, \text{ſin.}\, h$ ad $\text{ſin.}\, \varphi$, poſito ſcilicet $P = 1 - \frac{\text{ſin.}^2 \varphi}{2.2} - \frac{3\, \text{ſin.}^4 \varphi}{2.2.4.4} - \&c.$ Ut itaque utriuſque Caloris comparatio ad numeros transferatur, animadverto, arcum ſemidiurnum ſub Æquatore pro varia Solis declinatione aliquam licet parvam variationem ſubire; nam in Tabula praecedentis Diſquiſitionis §. 12. novies invenitur $h = 90^\circ\, 33'$, ſepties occurrit $h = 90^\circ\, 34'$, quinquies reperitur $h = 90^\circ\, 35'$, demum ter deprehenditur $h = 90^\circ\, 36'$; atque hinc medius ipſius h valor prodit circiter $90^\circ\, 34'$. Ecce jam calculi typus

$\varphi = 23^\circ\, 28'$ $\quad h = 90^\circ\, 34'$

$log.\ ſin.\ \varphi$	9,6001181
$log.\ ſin.^2\varphi$	9,2002362
$log.\ 4$	0,6020600
$log. \frac{ſin.^2\varphi}{2.2}$	8,5981762
$-\frac{ſin.^2\varphi}{2.2}$	$-$0,039644
$log.\ ſin.^4\varphi$	8,4004724
$log.\ 3$	0,4771213
$log.\ 3\ ſin.^4\varphi$	8,8775937
$log.\ 2.2.4.4$	1,8061800
$log. \frac{3\,ſin.^4\varphi}{2.2.4.4}$	7,0714137
$-\frac{3\,ſin.^4\varphi}{2.2.4.4}$	$-$0,001179
$log.\ ſin.^6\varphi$	7,6007086
$log.\ 3.3.5$	1,6532125
$log.\ 3.3.5\,ſin.^6\varphi$	9,2539211
$log.\ 2.2.4.4.6.6$	3,3624825
$log. \frac{3.3.5\,ſin.^6\varphi}{2.2.4.4.6.6}$	5,8914386

$\frac{3.3.5\,\text{ſin.}^6\varphi}{2.2.4.4.6.6}$	$-0,000078$
log. $\text{ſin.}^8\varphi$	6,8009448
log. 3.3.5.5.7	3,1972806
log. $3.3.5.5.7\,\text{ſin.}^8\varphi$	9,9982254
log. 2.2.4.4.6.6.8.8	5,1686625
log. $\frac{3.3.5.5.7\,\text{ſin.}^8\varphi}{2.2.4.4.6.6.8.8}$	4,8295629
$-\frac{3.3.5.5.7\,\text{ſin.}^8\varphi}{2.2.4.4.6.6.8.8}$	$-0,000007$

Eſt igitur $P = 1 - 0,039644 - 0,001179 - 0,000078$ $- 0,000007 = 0,959092$.

Quare

log. P	9,9818603
log. *ſin.* h	9,9999788
log. P *ſin.* h	9,9818391
P *ſin.* h	0,959045
ſin. φ	0,398215

Quapropter Calor Annuus Æquatorius ſe habet ad annuum Poli Calorem ut 959045 ad 398215, ſive ut 959 ad 398 vel proxime ut 17 ad 7, & ſemeſtris utrimque Calor ut 17 ad 14.

10. Formulae $\int dv' ſin. h \sqrt{(1 - ſin.^2 \varphi ſin.^2 v)}$ conſideratio elegantiſſimam ac prorſus miram patefacit Ellipſeos proprietatem, & cum annuo Solis Calore in regionibus aequatoriis analogiam atque conſenſum. Si enim ellipſis deſcribatur, cujus ſemiaxis major ſit unitati aequalis, excentricitas vero ſinum obliquitatis Eclipticae adaequet, & huic Ellipſi circulus circumſcribatur radium habens ſemiaxi majori aequalem; Ellipſeos perimeter ducta in ſinum arcus ſemidiurni ſub Æquatore, ſive in ſinum 90° 34' Calorem Æquatoris annuum repraeſentat, circuli vero circumſcripti peripheria ducta in Ellipſis excentricitatem Calorem annuum Polarem exprimit. Reapſe ſemiaxe
Fig. 3 majori $CP = 1$, excentricitate $FC = ſin.\,\varphi$ deſcribatur ſemiellipſis PEQ, & radio eodem PC ſemicirculus POQ, & producantur in O, & M ſemiaxis minor CE, & ordinata DN. Accepto arcu $OM = v$, ſeu Solis longitudini, oritur $DC = ſin.\ v$, $MD = cos.\ v = \sqrt{(1 - ſin.^2 v)}$. Eſt autem ex natura Ellipſis $EC = \sqrt{(PC^2 - FC^2)} = \sqrt{(1 - ſin.^2 \varphi)}$, & $OC : EC :: MD : ND$, ſive $1 : \sqrt{(1 - ſin.^2 \varphi)} :: \sqrt{(1 - ſin.^2 v)} : cos.\,\varphi\, cos.\, v$. Igitur $ND = cos.\,\varphi\, cos.\, v$. Capiatur jam differentiale ordinatae ND, quod eſt $-dv\, ſin.\, v\, cos.\,\varphi$, ſeu $-dv\, ſin.\ v \sqrt{(1 - ſin.^2 \varphi)}$, hujuſque quadratum $dv^2\, ſin.^2 v (1 - ſin.^2 \varphi)$, tum differentiale abſciſſae centralis

CD, quod invenitur $dv \cos. v$, ſive $dv \sqrt{(1 - ſin.^2 v)}$, & hujus quadratum $dv^2 (1 - ſin.^2 v)$: Summa hujuſmodi quadratorum oritur $= dv^2 (1 - ſin.^2 \varphi ſin.^2 v)$, & ejus radix quadrata praebet elementum arcus Elliptici *EN*; ac proinde detegitur arcus ipſe $EN = \int dv \sqrt{(1 - ſin.^2 \varphi ſin.^2 v)}$; in qua ſane expreſſione integrali ſi accipitur $v = 360°$, ſive = circuli peripheriae, abit *EN* in perimetrum totam Ellipſis. Igitur in hypotheſi $v = 360°$ ſe habet $\int dv ſin.h \sqrt{(1 - ſin.^2 \varphi ſin.^2 v)}$ ad $2\pi\ ſin.\varphi$, quemadmodum Calor annuus Æquatoris ad Calorem annuum Poli; nimirum ut praedictae Ellipſis perimeter ducta in *ſinum* 90.° 34ʹ ſe habet ad peripheriam circuli circumſcripti ductam in Ellipſis excentricitatem, ita Calor Æquatorius ad Polarem.

11. Liceat hic obiter animadvertere, ex evolutione quantitatis *P*, ſeu ſeriei $1 - \frac{ſin.^2 \varphi}{2.2} - \frac{3 ſin.^4 \varphi}{2.2.4.4} - \&c.$ (§. 9.) prodire rationem inter perimetrum Ellipſis *PEQ*, & peripheriam circuli *POQ*, quippe Ellipſis perimeter invenitur $= 2\pi P = 2\pi \times 0,959092$, & circuli circumſcripti peripheria $= 2\pi$. Igitur perimeter Ellipſis eſt ad circuli peripheriam uti $2\pi \times 0,959092$ ad 2π, ſive ut 959092 ad 1000000, vel ut 24 ad 25 fere.

12. Celeberrimus Geometra, & Aſtronomus To-

bias MAYERUS, ut Caloris Medii quantitatem non modo ſub Æquatore & Polo, ſed in omnibus Terrae locis & ſub quacumque latitudine definiret, formulam propoſuit ſimplicem oppido & elegantem, ad cujus formulae ductum Tabulas binas conſtruxit, quae medium Caloris gradum ſecundum Scalas Thermometri Reaumuriani, & Fahrenheitiani in variis latitudinibus repraeſentant. Praeſtat ipſum hac de re perſpicue diſſerentem audire: *Primum ergo conſtat* (inquit ille) *in unoquoque locorum terreſtrium certum definitumque regnare caloris gradum, qui medium tenet inter aeſtivum & hybernum, itemque inter diurnum & nocturnum, atque a climate, cui locus ſubjacet, ſive elevatione poli ejus loci pendet. Uti enim radii ſolares eo obliquiores in ſuperficiem terrae incidunt, quo magis ab aequatore ſunt remoti; ita minorem eſſe oportet hunc medium caloris gradum in locis majoris latitudinis. Facile autem oſtendi poteſt, ſi gradus medius ſecundum ſcalam thermometri cujuslibet ſub Æquatore ponatur* $= m$, *ſub polis vero* $= m - n$, *fore gradum medium ſub latitudine quavis* φ *proxime* $= m - n \sin.^2 \varphi$; *ſive differentiam inter calorem medium ſub aequatore & eum, qui loco latitudinis* φ *convenit, eſſe in ratione duplicata ſinus latitudinis. Hinc igitur, cognito gradu medio ſub duabus diverſis latitudinibus, licebit definire valores litterarum* m *&* n, *ac pro-*

inde gradum medium ſub alia quacumque latitudine (*a*).

13. Hiſce poſitis ſtatuit Mayerus gradum caloris medium in locis depreſſioribus Æquatori proximis eſſe gr. 24. in ſcala Thermometri Reaumuriani, ſub latitudine vero 49°, aut 50° eundem eſſe gr. 9 ſupra congelationis terminum, quantum quidem Mayero ex aliquali comparatione obſervationum thermometricarum ſub his latitudinibus inſtitutarum licuit colligere. Hinc vero deducitur aequatio $9 = m - n\,ſin.^2\varphi = 24 - n\,ſin.^2 49^\circ = 24 - \frac{9}{16}n$, ſive $n = 26\frac{2}{3}$, & hac ſubtracta a calore medio ſub Æquatore, ſeu a 24, oritur gradus caloris medius ſub Polis $= -2\frac{2}{3}$ infra congelationis limitem. At ob obſervationes neque ſumme accuratas, neque ſumma cura inter ſe comparatas, loco $-2\frac{2}{3}$ aſſumit Mayerus gradum medium ſub Polis $= 0$, talem nempe, quo aqua incipit congelari. Proinde aſſumpto tam m quam $n = 24$, oritur medius caloris gradus ſub latitudine quavis φ hac forma repraeſentatus $24\,(1 - ſin.^2\varphi) = 24\,cos.^2\varphi = 12 + 12\,cos.\,2\varphi$, quae ſequentem Tabulam largitur.

(*a*) Vid. Tobiae MAYERI *Opera Inedita* Vol. I. pag. 4. Gottingae 1775.

TABULA CALORIS MEDI ſ

Secundum Scalam Reaumurianam.

Latitudo Loci *Gr.*	*Gradus Therm.* *Reaum.*
0	24
5	23 ¾
10	23 ¼
15	22 ½
20	21 ¼
25	19 ¾
30	18
35	16
40	14
45	12
50	10
55	8
60	6
65	4 ¼
70	2 ¾
75	1 ½
80	8 ¾
85	0 ¼
90	0

14. Aſſumpta ſcala alia quacumque, quae datam habet rationem ad Reaumurianam, facilis negotii res erit ſecundum ſcalam hanc novam ex praemiſſa formula tabulam conſtruere Caloris medii ſub quacumque latitudine repraeſentatricem. Quum itaque in Scala Fahrenheitiana gradus 84 reſpondeant gradibus 24 Reaumurianae, & congelationis gradus in illa ſit 32, invenitur medius caloris gradus ſub latitudine φ hac forma expreſſus $84 - 52 \mathit{ſin.}^2 \varphi = 84 - 52 + 52 \mathit{cos.}^2 \varphi = 32 + 52 \mathit{cos.}^2 \varphi = 32 + 26 + 26 \mathit{cos.} 2\varphi = 58 + 26 \mathit{cos.} 2\varphi$, ad cujus ductum ſequens contexitur

Tabula Caloris Medii

Secundum Scalam Fahrenheitianam

Latitudo Loci Gr.	*Gradus Therm.* *Fahrenheit.*
0	84
5	83 ½
10	82 ½
15	80 ½
20	78
25	74 ½

Latitudo Loci Gr.	*Gradus Therm. Fahrenheit.*
30	71
35	67
40	62 ½
45	58
50	53 ½
55	49
60	45
65	41 ½
70	38
75	35 ½
80	33 ½
85	32 ½
90	32

15. *Atque hic eſt*, inquit MAYERUS, *gradus caloris medius loci cujuſcumque loco medio planetarum in aſtronomicis non inepte comparandus. Quemadmodum autem Aſtronomi ſolent huic loco medio plures aequationes adplicare, atque adeo magis magiſque ad veram adpropinquare: ita ſimili quoque modo correctiones, quibus gradus caloris medius ad verum poſſit reduci, inveſtigandae ſunt.* Quum itaque calor ab ima terrae ſuperficie aſcendendo ſenſim ſenſimque minuatur ad certam uſ-

que altitudinem, ultra quam incipit rurſus augeri, hocque caloris in majoribus altitudinibus decrementum tum montes vel in ipſa Zona Torrida perpetua nive geluque rigentes, tum notiſſimae Gallorum Mathematicorum obſervationes in Peruvianis montibus prope Æquatorem inſtitutae aperte demonſtrent; inde correctionem praecipuam Mayerus deduxit ab altitudine loci petitam, poſuitque interim caloris gradum a ſuperficie maris aſcendendo decreſcere in ea ratione qua augetur altitudo: quo nixus fundamento hanc facillimi uſus regulam condidit: „ Si data elevatione loci ſupra maris ſuperficiem ejuſque latitudine geographica inveſtigandus ſit gradus caloris illius loci medius, quaeratur primum ad datam latitudinem gradus medius ſecundum ſcalam Reaumurianam ſive per formulam ſupra traditam, ſive ex tabula ibidem expoſita; deinde altitudinis loci in hexapedis Pariſinis expreſſae pars centeſima, quae correctionem hanc primam exhibet, ab illo gradu auferatur, reſiduum erit gradus medius dati loci quaeſitus “.

16. Verum dolendum eſt, ingenioſiſſimam Mayeri methodum non uſque adeo generalem eſſe, ut omnibus terrae locis climatibuſque conveniat; nam adnotante Mayerani operis editore Lichtenbergio conſenſus inter altitudines medias Thermometri obſervatas, &

computatas magnus utique eſt in toto illo terrarum tractu, qui parallelum Promontorii Bonae Spei, & Holmienſem, nec non meridianum Holmienſem, & Mexicanum interjacet: Hiſce vero limitibus praetergreſſis formula Mayeriana a veritate longe aberrat, ita ut altitudines mediae Thermometri PETROPOLI & WILNAE v. c. obſervatae deficiant, illa gradibus 8, 5; haec 10, 3 ab ea, quam Mayeriana formula praebet.

17. Caeterum in inventione formulae $12 + 12 \cos. 2\varphi$ videtur MAYERUS eo praeſertim reſpexiſſe, ut calor medius ſub Æquatore *Maximus*, ſub Polis *Minimus* inde conſequeretur; cui quidem conditioni formula ipſa ſatisfacit, nam ejus differentiale $- 24\, d\varphi\, \mathit{ſin.}\, 2\varphi$ nihilo aequatum duos praebet latitudinis φ valores, alterum $\varphi = 0$, alterum $\varphi = 90°$, quorum prior Æquatorem, poſterior Polum deſignat. Ex priori autem τȢ φ valore haberi Maximum, ex ſecundo Minimum vel inde patet, quod ſecunda differentia $- 48\, d\varphi^2 \cos. 2\varphi$ negativa ſit, ſeu $- 48\, d\varphi^2$ ſumpto $\varphi = 0$, & contra affirmativa, ſcilicet $+ 48\, d\varphi^2$ facto $\varphi = 90°$. Verum hujuſmodi conditionis complementum, quo niti Mayeriana formula videtur, pro accurato formulae ipſius examine haberi nullo modo poteſt, quum alia multo generalior $a + b \cos. 2\varphi + c \cos. 4\varphi + d \cos. 6\varphi +$ &c. eandem conditionem aeque impleat.

18. Inventa jam in praecedenti Diſquiſitione formula $2\,ſin.\,h\,cos.\,\lambda\,cos.\,\delta \pm 2h\,ſin.\,\lambda\,ſin.\,\delta$ exprimente Calorem Solis Diurnum pro qualibet Solis declinatione, & latitudine terreſtri; & cautione adhibita uſurpandi ſecundum formulae terminum cum ſigno affirmativo ſi declinatio & latitudo ejuſdem nominis fuerint, cum negativo ſi diverſi; oportet nunc Calorem Solis Annuum non ſub Æquatore dumtaxat, & Polo, quod antea praeſtitimus, ſed in quacumque elevatione Poli, ſeu in quolibet Telluris loco ſupputare ac definire. Utar ad hanc rem eximia Newtoni methodo *ducendi curvam geometricam per data quotcumque puncta*, quam Newtonus ad Leibnitium ſcribens merito inter inventa pulcherrima numerabat, quamque ſummus Geometra Rogerius Cotesius inſigni Opuſculo *De Methodo Differentiali Nevvtoniana* elegantiſſime demonſtravit auxitque. Sumantur itaque in quavis Terrae regione dies bini dimidiato anno a ſe invicem diſtantes, in quibus idcirco eadem erit declinatio Solis, ſed ejuſdem nominis cum latitudine loci pro die uno, diverſi nominis pro die altero oppoſito; vocentur h, h' arcus dierum ipſorum ſemidiurni: eritque calor diei illius, quo Sol declinationem habet cognominem latitudini, $2\,ſin.\,h\,cos.\,\lambda\,cos.\,\delta + 2\,h\,ſin.\,\lambda\,ſin.\,\delta$, calor diei oppoſiti $2\,ſin.\,h'\,cos.\,\lambda\,cos.\,\delta - 2\,h'\,ſin.\,\lambda\,ſin.\,\delta$, & calor

dierum binorum ſimul $= 2\cos.\lambda\cos.\delta\,(\sin.h + \sin.h')$ $+ 2\sin.\lambda\sin.\delta\,(h - h')$. Quum autem ex Trigonometriae formulis habeatur $\sin.h + \sin.h' = 2\sin.\frac{h+h'}{2} \times \cos.\frac{h-h'}{2} = 2\sin.\frac{180°}{2}\cos.\frac{h-h'}{2} = 2\cos.\frac{h-h'}{2}$; propterea calor binorum dierum oppoſitorum fiet $= 4\cos.\lambda\cos.\delta\cos.\frac{h-h'}{2} + 4\left(\frac{h-h'}{2}\right)\sin.\lambda\sin.\delta$. Hic porro calor ſi diebus ſingulis ab aequinoctio uſque ad ſolſtitium ad hujus praeſcriptum formulae rite ſupputetur, aſſequimur calorem ſemiannuum, cujus proinde duplum dat calorem integrum anni unius; in quaeſtione etenim adeo abſtruſa, ac tot undique ſepta difficultatibus poſtulati loco aſſumere licet, calorem Solis duabus oppoſitis anni tempeſtatibus aggregatum eundem eſſe cum illo, qui binis aliis tempeſtatibus oppoſitis accumulatur. Sed quoniam haec ſupputatio ſemel & nonageſies repetita laboris eſſet ac moleſtiae pleniſſima, operae pretium erit, ſummam illam caloris, qui ab aequinoctio verno ad ſolſtitium aeſtivum, & ab aequinoctio autumnali ad ſolſtitium hybernum acervatur, per approximationem inquirere. Itaque ex Aſtronomicis manifeſtum eſt, in quolibet Terrae loco data Solis longitudine haberi declinatio-

nem δ ex analogia, uti radius ad ſinum longitudinis Solis, ita ſinus obliquitatis Eclipticae ad ſinum declinationis, ſeu ad *ſin.* δ: inventa porro declinatione Solis aſſequimur differentiam aſcenſionalem $\frac{h-h'}{2}$ ex altera analogia uti radius ad tangentem latitudinis, ita tangens declinationis ad ſinum differentiae aſcenſionalis. Eapropter aſſumpta Solis longitudine graduum o, 15°, 30°, 45°, &c. ac poſita Eclipticae obliquitate $= 23°\frac{1}{2}$ (in hac quippe quaeſtione negligimus minutias), pro Urbe cujus latitudo foret $51°\frac{1}{2}$ cujuſmodi eſſet Gottinga, ſequens ſtruitur

T A B E L L A

Solis Longitudo	*Declinatio ſeu* δ	*Differentia Aſcenſionalis ſeu* $\frac{h-h'}{2}$
o	o	o
15°	5°. 55'	7°, 483
30°	11°. 31'	14°, 816
45°	16°. 23'	21°, 68
60°	20°. 12'	27°, 55
75°	22°. 39'	31°, 63
90°	23°. 30'	33°, 13

19. In tertia hujus tabulae columna differentia aſcenſionalis exhibetur per gradus, parteſque gradus decimales ad majorem ſubſequentis calculi facilitatem. Inventa jam pro latitudine 51° ½ tum declinatione Solis tum differentia aſcenſionali ſingulis quindecim gradibus longitudinis Solis reſpondente conſtruitur altera

TABELLA

Solis Longitudo	*Productum* $\cos.\delta\ \cos.\frac{h-h'}{2}$	*Productum* $\frac{h-h'}{2}$ *ſin.* δ
0	1, 0000	0
15°	0, 9862	0, 7723
30°	0, 9473	2, 9537
45°	0, 8915	6, 1128
60°	0, 8321	9, 5138
75°	0, 7857	12, 1827
90°	0, 7505	13, 2107

ubi producta tertiae columnae fiunt multiplicando valorem quemvis *ſin.* δ per reſpondentem differentiam aſcenſionalem qualis exhibetur in tertia primae tabulae columna. Sed quoniam in producto $\left(\frac{h-h'}{2}\right)$ *ſin.* δ differentia aſcenſionalis $\frac{h-h'}{2}$ accipienda eſt non gra-

duum numero expreſſa, ſed radii partibus repraeſentata, ut per ſe palam eſt; iccirco hujus tertiae columnae numeri reductionem ſubibunt, quam poſtmodum indicabimus. Interea ſpectentur numeri ſecundae columnae inſtar ſeptem aequidiſtantium ordinatarum ad Curvam generis parabolici, cujus area extremis interjecta exhibetur per formulam $\frac{41 A + 216 B + 27 C + 272 D}{840} R$ (*a*). In hac vero formula deſignat A ſummam ordinatarum extimarum, hoc eſt ſummam primae, & ultimae $= 1,0000 + 0,7505 = 1,7505$; B ſummam proximarum extimis, hoc eſt ſummam ſecundae & penultimae $= 0,9862 + 0,7857 = 1,7719$; C ſummam ſequentium, nempe tertiae & antepenultimae $= 0,9473 + 0,8321 = 1,7794$; D quartam ordinatam $= 0,8915$: intervallum denique inter extimas ordinatas, nimirum areae quadrandae baſis indigitatur per litteram $R = 91$, quotus ſcilicet eſt dierum numerus ab aequinoctio ad ſolſtitium. Ducta itaque hac area in $4 \cos. \lambda$, numerica ſupputatio dabit $4 Ar. \times \cos. \lambda = 200,56$. Idem fiat in numeris tertiae columnae, per quos rurſus ſeptem aequidiſtantes ordinatae intelliguntur repraeſentari, fitque iterum $A = 0 + 13,2107 = 13,2107$; $B = 0,7723 + 12,1827 = 12,9550$; $C = 2,9537 + 9,5138 = 12,4675$; $D = 6,1128$;

(*a*) Vid. COTES. Opuſc. cit. Prop. VI.

$R = 91$. Area hinc per formulam Coteſianam elicienda multiplicari debet per longitudinem arcus gradus unius, hoc eſt per o, 01745, eo quod in numeris tertiae columnae factorem $\frac{h - h'}{2}$ involventibus fieri debet converſio factoris ipſius de numero graduum in partes radii per unitatem deſignati: quod quidem aſſequimur per vulgatiſſimam analogiam

1° : n° : : o, 01745 : ad quartum $= 0,01745 \times n$, qui longitudinem exprimit arcus numerum n graduum continentis. Ducta autem quantitate $0,01745 \times Ar'$. per $4 \mathit{ſin}. \lambda$, invenietur $4 \times 0,01745 \times Ar'. \times \mathit{ſin}. \lambda = 31,96$. Igitur $4\, Ar. \times \cos. \lambda + 4 \times 0,01745 \times Ar'. \times \mathit{ſin}. \lambda$ $= 200,56 + 31,96 = 232,52$; atque hujus duplum 465, 04 exhibet calorem annuum integrum pro elevatione Poli 51° ½, cujuſmodi eſt Gottingae, aut Londini.

20. Sub Æquinoctiali Circulo, ubi $\lambda = 0$, $\frac{h - h'}{2} = 0$, Caloris quantitas binis diebus ſemeſtri ſpatio diſtantibus collecta $4 \cos. \lambda \cos. \delta \cos. \frac{h - h'}{2} + 4 \left(\frac{h - h'}{2}\right) \mathit{ſin}. \lambda\, \mathit{ſin}. \delta$ abit in $4 \cos. \delta$. Igitur ſummam omnium $\cos. \delta$ oportet invenire. Eſt autem

Longitudo Solis	*Declinatio δ*	*cos. δ*
0	0°	1, 0000
15°	5°. 55'	0, 9946
30°	11°. 30'	0, 9799
45°	16°. 23'	0, 9594
60°	20°. 12'	0, 9385
75°	22°. 39'	0, 9229
90°	23°. 30'	0, 9170

Quapropter $A = 1,9170$; $B = 1,9175$; $C = 1,9184$; $D = 0,9594$; $R = 91$; quibus valoribus in Coteſiana formula ſubſtitutis oritur Area $= 88,48$: haec autem ducta in 8 exhibet calorem annuum aequinoctialem $= 707,8$. Igitur calor Solis annuus ſub Æquatore ſe habet ad calorem annuum LONDINI, vel GOTTINGAE uti 708 ad 465, hoc eſt uti 236 ad 155 quamproxime.

21. Quum in praecedenti Diſquiſitione Calor diurnus ſub Polo inventus ſit $= 2\pi$ *ſin.* $\delta = 6,2832$ *ſin.* δ, ſumpto videlicet π pro circuli ſemiperipheria; aſſequemur hac eadem methodo calorem Poli ſemeſtrem, vel ſi mavis annuum (altero quippe ſemeſtri nullus ob Solis abſentiam producitur calor), ſi ſumma omnium *ſin.* δ ab aequinoctio ad proximum ſolſtitium accipia-

tur, eaque ducta in 6, 2832 duplicetur. Ita vero habetur

Longitudo Solis	*Declinatio* δ	*ſin*
0	0	0
15°	5°. 55'	0, 1031
30°	11°. 30'	0, 1993
45°	16°. 23'	0, 2821
60°	20°. 12'	0, 3453
75°	22°. 39'	0, 3851
90°	23°. 30'	0, 3987

Quamobrem $A = 0,3987$; $B = 0,4882$; $C = 0,5446$; $D = 0,2821$; $R = 91$: atque hiſce valoribus in Coteſiana formula ſubrogatis, prodit area parabolica, quae multiplicata per $2 \times 6,2832$ patefacit ſemeſtrem quaeſitum Poli calorem $= 2 \times 6,2832 \times Ar. = 269,5$. Eſt igitur

Calor Annuus

Æquatorius uti	7078
Londinenſis, vel Gottingenſis	4650
Polaris	2695

22. Sit modo invenienda Caloris Annui menſura in Urbe Papiensi, in qua ſcribimus, cujus latitudo, uti notum eſt, ad 45° : 11' pertingit. Eſt igitur

Solis longitudo	*Declinatio δ*	*Differentia aſcenſionalis* $\frac{h - h'}{2}$	*Diff. Aſcenſ. in partibus radii*
0	0	0	0
15°	5° : 55'	4° : 13'	0,073594
30°	11° : 31'	8° : 18'	0,144861
45°	16° : 23'	12° : 2'	0,210020
60°	20° : 12'	15° : 7'	0,263835
75°	22° : 39'	17° : 14'	0,300777
90°	23° : 30'	17° : 58'	0,313576

Ex hiſce porro inveniuntur producta $\cos.\delta \cos. \frac{h - h'}{2}$, & $\frac{h - h'}{2} \mathit{ſin}.\delta$, uti ſequens Tabella demonſtrat

Solis longitudo	*Productum* $\cos.\delta \cos. \frac{h - h'}{2}$	*Productum* $\frac{h - h'}{2} \mathit{ſin}.\delta$
0	1, 00000	0
15°	0, 99208	0, 00757

Solis longitudo	*Productum* $\cos.\delta \cos.\frac{h-h'}{2}$	*Productum* $\frac{h-h'}{2}$ *ſin.*
30°	0, 96960	0, 02892
45°	0, 93831	0, 05924
60°	0, 90602	0, 09110
75°	0, 88144	0, 11583
90°	0, 87234	0, 12509

Hinc uſurpata, ut prius, Coteſiana formula, ſumptiſque numeris ſecundae, & tertiae columnae pro ordinatis binarum Curvarum generis parabolici aſſequimur

Pro Curva prima	*Pro Curva altera*
$A = 1,87234$	$A = 0,12509$
$B = 1,87352$	$B = 0,12340$
$C = 1,87562$	$C = 0,12002$
$D = 0,93831$	$D = 0,05924$
$R = 91$	$R = 91$
4. *Area* $\times \cos.\lambda = 240, 13$	4. *Areà* $\times$ *ſin.* $\lambda = 15, 719$

Quamobrem 4. *Area* $\times \cos.\lambda +$ 4. *Areà* $\times$ *ſin.* $\lambda = 255, 85$,

hujuſque duplum 511,70 ſuppeditat quaeſitam Caloris annui Solaris menſuram ſub elevatione Poli 45° : 11', qualem PAPIÆ agnovimus: Erit igitur

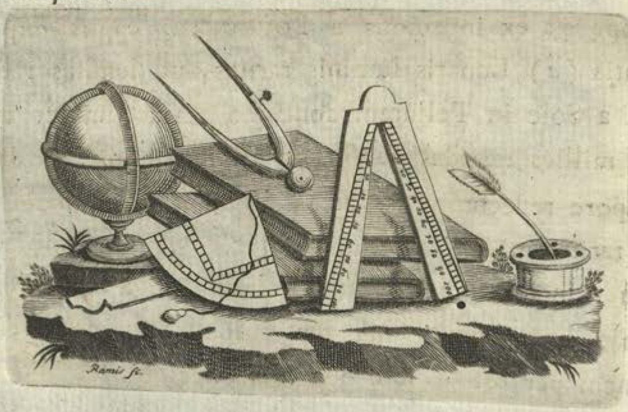
Calor Solaris Annuus

Æquatorius	7078
Papienſis	5117
Londinenſis, vel Gottingenſis . . .	4650
Polaris	2695

23. Hactenus tacite poſuimus, Calorem Solis ab ejus radiis in Tellurem irruentibus unice pendere, eſſeque Caloris ipſius intenſionem numero radiorum ubique proportionalem: neque hanc aſſumptionem infirmat Lunaris lux, quae Plenilunii tempore in foco maximorum ſpeculorum collecta ne minimum quidem caloris excitat ſenſum, ſummeque dilatabili thermometri exquiſitiſſimi liquori expandendo impar eſt. Ea quippe eſt ex ingenioſis a BOUGUERIO captis experimentis (*a*) Lunaris luminis raritas, ut denſitas radiorum a Sole in Tellurem continuo emiſſorum ter centies millies excedat denſitatem illorum, qui Plenilunii tempore reflectuntur a Luna. Hanc ipſam Lunaris lucis raritatem ſuadet ratiocinatio geometrica firmiſſimis nixa principiis: nam quum Solaris lux a colluſtrato Lunae hemiſphaerio ita repercutiatur, ut poſt reflexionem per totam fere ſuperficiem globi ſemidiame-

(*a*) Opt. Lib. I. Sect. II. Art. XI.

trum habentis diſtantiae Lunae a Tellure aequalem diffundatur, ſitque adeo ea ſuperficies ad ſuperficiem globi Lunaris ut 46249 ad 1 (accepta nimirum ſemidiametro Lunae apparenti 16′, quae ad ſemidiametrum ejus veram ſe habet ut 465 ad 100000); erit iccirco media luminis denſitas a Sole ad Lunam, vel etiam ad Terram directe propagati tot vicibus major denſitate luminis a Luna in Terram repercuſſi (ſi vel totum, quod eſſe nequit, id lumen reflecteretur), quot vicibus excedit ampliſſima illa ingentis ſphaerae ſuperficies aream circuli Lunaris maximi, ſeu quartam Lunaris ſuperficiei partem, quae pars menſuram luminis ante reflexionem repraeſentat, nimirum 184996 vicibus. Quapropter minime erit mirandum, quod tanta Lunaris lucis attenuatio & raritas cum nullo ſit calore conjuncta.



DISQUISITIO III.

De Sanguinis Restitutione hujusque Problematis affinitate, et analogia cum Problemate anticipationis, seu pecuniae in antecessum numeratae.

1. Illuſtris Iatromathematicus Jacobus KEILLIUS in *Tentaminibus Medico-Phyſicis*, Tentamine I. *De Sanguinis quantitate* determinaturus ſanguinis redintegrationem, quae ex quotidiano victu continentem partium interitum reparante proficiſcitur, calculum inaedificat fundamento undique labanti & ruinoſo: comparat ſcilicet Problema hoc cum altero Algebrae Scriptoribus uſitato, quo quaeritur vini quantitas poſt aliquem dierum numerum in dolio ſuperſtes, ſi certa vini menſura quotidie exhauriatur ac tantundem aquae ſemper infundatur. *Sit itaque victus*, inquit citato loco KEILLIUS, *quem ad ſuſtinendas vires, & ad quotidianos partium interitus compenſandos recipimus, librarum quatuor. Hic*

cibus cum ſanguine commixtus, & quaſi in unum concoctus, cum eo ſimul in glandulis ſecernetur: ita ut novus, vetuſque uno eodemque tempore amandentur: & primo ſecernantur in ratione quantitatibus proportionali: Quaeritur jam, quantum veteris ſanguinis, poſt datum temporis ſpatium in corpore relinquetur. Haec quaeſtio eadem eſt, ac ſi vas aliquod 200. congiis vini repletum eſſe ponamus, & inde quatuor quotidie exhauſtos, totidemque aquae infuſos, ut vas plenum ſemper relinquatur. Quaeritur jam quantum vini poſt aliquem dierum numerum in vaſe remanebit. Hoc porro inſtabili ac lubrico nixus principio KEILLIUS calculum ponit a veritate longe aberrantem, ſed eleganti quadam veri ſpecie, & quaſi fuco minus attentis & ſagacibus imponentem. Itaque quum de quaeſtione agatur inter Phyſiologiae Scriptores celeberrima ac nobiliſſima, de ſanguinis nempe reparatione ac mutatione, operae praetium judico rem ex integro aggredi, & Keilliano convulſo everſoque fundamento novum ponere calculum minus quidem, quam Keillianus, facilem & expeditum, ſed nec minus elegantem, & veritati rerumque naturae magis conformem.

2. Jam in primis penes Phyſiologorum commendatiſſimos in confeſſo eſt, animalis corporis ſanguinem intra diei ſpatium inſenſibili perſpiratione avolantem, ſingulis momentis de corpore dece-

dere, ſingulisque momentis ſanguinem novum ex alimentis elaboratum accedere veteri, & jacturas momentaneas reparare. Abſurda igitur eſt comparatio ex vaſe vinario petita, ex quo non jugiter & continenter, ſed ſimul & ſemel in die certa vini quantitas exhauritur, & aquae infunditur: ut aequa eſſet & numeris omnibus perfecta comparatio neceſſe foret dolium imaginari, ex cujus oſtiolo ſeu foramine vinum continenter effluerat, & per foramen alterum aequale aqua perennis eodem impetu ſimul influeret. Tunc vero poſtulare quantum vini poſt datum temporis intervallum in dolio futurum ſit, & contra quanto tempore vinum ad datam menſuram decreſcet, idem prorſus eſt, ſi caetera paria accipiantur, ac petere quantum veteris ſanguinis poſt aliquem dierum numerum in humano corpore reliquum ſit, vel contra quot diebus Sanguis vetus ad datam uſque particulam redigetur.

3. In hoc itaque ſolvendo Problemate calculus informandus erit in hunc modum:

Sit animalis Sanguinis quantitas . . . . $= a$
Pars ejus infiniteſima ſingulis momentis avolans $= p$
Numerus momentorum in die naturali . . . $= n$
Quantitas *mixti* Sanguinis intra diem decedens $= b = np$
Ergo Sanguis vetus in fine primi momenti $= a - p$

Evidens eſt, ſubſequentem calculum eodem modo

rite procedere, etiamſi reparatio Sanguinis non toto diei tempore, ſed paucis intra diem horis abſolvatur, eo ſcilicet intervallo quo Chyloſis, & Haematoſis perficitur; ſiquidem in hac hypoteſi Haematoſis diurnae tempus dividitur in momentorum numerum, omniaque fiunt quemadmodum in prima hypotheſi. Caeterum de hac continenti ac perpetua Sanguinis jactura, & reparatione perbelle diſſerit Facultatis Medicae Pariſienſis Socius ROBERTUS in recentiſſimo Opuſculo *De Senectute*. *La nature* (inquit ille) *y pompe* (ſcilicet in Athmoſphaera) *& en attire l'eſpece d'humidité, dont elle a beſoin pour renouveller cette fumée aqueuſe qui s'évapore de toute la ſurface du corps par forme de tranſpiration. Cette action de pomper, & d'attirer l'humidité, eſt une faculté de la peau, qu'on ne peut pas révoquer en doute; l'exemple des Bouchers, des Chaircuitiers, des Cuiſiniers, donne toutes les preuves que l'on pourroit déſirer à cet égard. Ils ont la plupart, ainſi que leurs femmes, la peau blanche, le teint frais & fleuri, & en général, beaucoup d'embonpoint; en un mot ils ſe nourriſſent par la peau. Mais ſi la nature pompe dans l'athmoſphére un ſuc alimentaire, l'on peut bien croire auſſi qu'elle y puiſe l'eau dont elle a beſoin pour s'humecter & ſe rafraichir.*

Quum autem ſecundo momento iterum decedat

pars minima p Sanguinis mixti ex vetere $a-p$, & ex novo adveniente, ac prioris jacturam reparante; ut inveniatur Sanguis vetus reſiduus in fine ſecundi momenti, fiat $a : p :: a-p : \frac{p(a-p)}{a}$, eritque $\frac{p(a-p)}{a}$ pars veteris Sanguinis in fine ſecundi momenti amiſſa.

Quare Sanguis vetus in fine ſecundi momenti reſiduus invenitur $= a-p-\frac{p(a-p)}{a} = \frac{(a-p)^2}{a}$.

Rurſus detegitur ſanguis vetus in fine tertii momenti ſuperſtes, ſi ex analogia $a : p :: \frac{(a-p)^2}{a} : \frac{p(a-p)^2}{a^2}$ eliciatur pars $\frac{p(a-p)^2}{a^2}$ veteris ſanguinis in fine tertii momenti decedens, eaque dematur a Sanguine veteri $\frac{(a-p)^2}{a}$, unde oritur $\frac{(a-p)^2}{a} - \frac{p(a-p)^2}{a^2}$ $= \frac{(a-p)^3}{a^2}$. Sicque porro Sanguis vetus in fine quarti momenti reperitur $\frac{(a-p)^4}{a^3}$; ac denique Sanguis vetus in fine momenti n:^{mi}, ſeu integrae diei ſuperſtes prodibit $= \frac{(a-p)^n}{a^{n-1}} = a - np + \frac{n(n-1)}{2a} p^2$

$$-\frac{n(n-1)(n-2)}{2.3a^2}p^3+\frac{n(n-1)(n-2)(n-3)}{2.3.4a^3}p^4$$

$$-\frac{n(n-1)(n-2)(n-3)(n-4)}{2.3.4.5a^4}p^5+$$ &c. Quum autem in hujuſmodi expreſſione coefficientium factores $n(n-1)(n-2)(n-3)$ - - - - abeant in $nnnn\ldots$, quandoquidem prae valore ipſius n infinito evaneſcunt numeri 1, 2, 3, &c.; propterea Sanguis vetus in fine unius diei ſuperſtes repraeſentatur per formulam

$$a-np+\frac{n^2p^2}{2a}-\frac{n^3p^3}{2.3a^2}+\frac{n^4p^4}{2.3.4a^3}-\frac{n^5p^5}{2.3.4.5a^4}$$

$$+\text{ \&c.}=a-b+\frac{b^2}{2a}-\frac{b^3}{2.3a^2}+\frac{b^4}{2.3.4a^3}$$

$$-\frac{b^5}{2.3.4.5a^4}+$$ &c. Atque hinc liquet, quam parum veritati conſonum ſit illud Keillianae hypotheſis conſectarium, quo ſanguis vetus in fine primae diei reſiduus ponitur $=a-b$. Ut modo inveniatur, quot diebus perficiatur ſanguinis inſtauratio, ita ut nimirum ſanguis vetus in corpore reſiduus quantitatem valde parvam ω adaequet, voco x numerum dierum quaeſitum, ac protinus nanciſcor $\frac{(a-p)^{nx}}{a^{nx-1}}=\omega$, ſeu

$\left(\frac{a-p}{a}\right)^{nx}=\frac{\omega}{a}$; hinc habeo x $log.$ $\left(\frac{a-p}{a}\right)^n$

$= \log. \omega - \log. a$, ſive $x \log. \left(1 - \frac{b}{a} + \frac{b^2}{2a^2} - \frac{b^3}{6a^3} + \frac{b^4}{24a^4} - \frac{b^5}{120a^5} + \&c.\right) = \log. \omega - \log. a$; & conſequenter $x = \frac{\log. \omega - \log. a}{\log. \left(1 - \frac{b}{a} + \frac{b^2}{2a^2} - \frac{b^3}{6a^3} + \frac{b^4}{24a^4} - \&c.\right)}$

Sit igitur juxta aſſumpta Keilliana $a = 20$ libr., $b = 4$ libr., $\omega = 0,0247$ libr., eritque $\log. \omega - \log. a = -1,6073030 - 1,3010300 = -2,9083330$, & $\log. \left(1 - \frac{b}{a} + \&c.\right) = \log. \left(1 - \frac{1}{5} + \frac{1}{50} - \frac{1}{750} + \frac{1}{15000}\right)$ ſatis proxime, ſeu $= \log. \frac{12281}{15000} = -0,0868576$. Quamobrem reperitur $x = \frac{-2,9083330}{-0,0868576} = \frac{29083330}{868576} = 33\frac{1}{2}$ diebus circiter, ſeu tribus diebus cum dimidio plus quam a Keillio proponitur. Idem invenitur brevius hoc pacto: Conſtat, logarithmum hyperbolicum quantitatis $1 - \frac{b}{a} + \frac{b^2}{2a^2} - \frac{b^3}{6a^3} + \frac{b^4}{24a^4} - \&c.$ eſſe $-\frac{b}{a}$, & logarithmum Tabularem, ſeu Briggianum aequari hyperbolico diviſo per 2, 302585; hinc prodibit

$$log.\left(1-\frac{b}{a}+\frac{b^2}{2a^2}-\&c.\right)=-\frac{b}{2,302585a}$$

$$=-\frac{1}{11,512925}=-0,08685 \text{; atque ideo } x=33\tfrac{1}{?}$$

diebus.

4. Ponatur præterea cum KEILLIO totum corpus libras 160. ponderare, & quaeratur quantum veteris corporis exacto demum anno relinquatur, ſi quotidianus partium fluidarum ſolidarumque interitus quatuor librarum aſſumatur. Ex ſuperiori formula deducitur illico $log.\,\omega = x\,log.\left(1-\frac{b}{a}+\frac{b^2}{2a^2}-\&c.\right)+log.\,a$, & capto $x=365$ diebus, $a=160$ libr., $b=4$ libr. oritur $log.\,\omega=-\frac{365b}{2,302585a}+2,20412=-3,9629373$ $+2,20412=-1,7588173$; proindeque $\omega=0,01742$ libr; quod iterum differt a Keilliano.

S C H O L I O N I.

5. Mira prorſus eſt hujus Problematis cognatio & affinitas cum Problemate *anticipationis*, ſeu pecuniae in anteceſſum numeratae: quaeritur ibi quantum datae ſummae vel ſortis *a* poſt annum elapſum relinquatur, ſi aſſumpto annuo ipſius foenore *b* detrahatur a

ſorte ſingulis momentis pars ſoenoris proportionalis; ſeu quod idem eſt, quantum Creditori debetur, ſi numerata in anteceſſum pecunia ipſe conſentiat in compenſationem ſingulis momentis faciendam partis proportionalis annuae uſurae. Enimvero diſpertito anno in infinitum momentorum numerum n, palam eſt, fore $\frac{b}{n}$ foenus primo momento reſpondens, & $a - \frac{b}{n}$ eſſe ſortem elapſo primo momento reſiduam. Porro ſi ſors a primo momento foenus dat $\frac{b}{n}$, ſors $a - \frac{b}{n}$ ſecundo momento foenus praebebit $\frac{\left(a - \frac{b}{n}\right)\frac{b}{n}}{a}$, quod a ſorte $a - \frac{b}{n}$ ſubtractum relinquit $a - \frac{b}{n} - \frac{\left(a - \frac{b}{n}\right)\frac{b}{n}}{a}$ $= \frac{\left(a - \frac{b}{n}\right)^2}{a}$ ſorti elapſo ſecundo momento reſiduae.

Inſtituta rurſus analogia $a : \frac{b}{n} :: \frac{\left(a - \frac{b}{n}\right)^2}{a} : \frac{\left(a - \frac{b}{n}\right)^2 \frac{b}{n}}{a^2}$, quartus proportionalis $\frac{\left(a - \frac{b}{n}\right)^2 \frac{b}{n}}{a^2}$ uſuram exprimit

tertio momento caducam, & $\frac{\left(a-\frac{b}{n}\right)^2}{a} - \frac{\left(a-\frac{b}{n}\right)^2 \frac{b}{n}}{a^2}$

$= \frac{\left(a-\frac{b}{n}\right)^3}{a^2}$ ſortem repraeſentat exacto tertio momento ſuperſtitem: ſicque porro exacto momento n: eſimo ſeu anno integro invenietur ſors reliqua

$= \frac{\left(a-\frac{b}{n}\right)^n}{a^{n-1}}$. Evoluta hac quantitate invenitur

$a - b + \frac{b^2}{2a} - \frac{b^3}{6a^2} + \frac{b^4}{24a^3} - \frac{b^5}{120a^4} +$ &c. habita ſcilicet ratione valoris infiniti n, prae quo in factore quolibet $n-1$, $n-2$, $n-3$, &c. evaneſcunt numeri -1, -2, -3, &c. Eſt autem haec formula plane eadem, quam ſupra pro ſanguinis renovatione deteximus, & formulae valor nihil aliud eſt niſi factum ex quantitate a in numerum, cujus logarithmus hyperbolicus eſt $-\frac{b}{a}$.

6. Haud diſſimili ratiocinatione oſtendi poteſt decantatum Problema, a Jac. BERNOULLIO in Actis Lipſienſibus propoſitum, & ſuppreſſa demonſtratione ſolutum. Problema Bernoullianis verbis expreſſum tale eſt:

Quaeritur ſi Creditor aliquis pecuniam foenori exponat ea lege, ut ſingulis momentis pars proportionalis uſurae annuae ſorti annumeretur, quantum ipſi finito anno debeatur. Ait Bernoullius pecuniam elapſo anno Creditori debitam aequari facto ex data ſorte in numerum, cujus logarithmus hyperbolicus eſt ratio foenoris ad ſortem.

7. Ad hoc demonſtrandum dicatur a ſors, b annua uſura, n numerus momentorum infinitus in anno: erit $\frac{b}{n}$ uſura primi momenti, & $a+\frac{b}{n}$ ſors in fine illius momenti. Dic jam, ſi ſors a momento primo foenus parit $\frac{b}{n}$, ſors $a+\frac{b}{n}$ gignet momento ſecundo uſuram $\frac{\left(a+\frac{b}{n}\right)\frac{b}{n}}{a}$, quae praecedenti ſorti $a+\frac{b}{n}$ annumerata ſortem dat in fine ſecundi momenti $\frac{\left(a+\frac{b}{n}\right)^2}{a}$. Tertio momento, propter analogiam $a:\frac{b}{n}::\frac{\left(a+\frac{b}{n}\right)^2}{a}:\frac{\left(a+\frac{b}{n}\right)^2\frac{b}{n}}{a^2}$, habetur foenus reſpondens $\frac{\left(a+\frac{b}{n}\right)^2\frac{b}{n}}{a^2}$,

quod rurſus ſorti praecedenti adjunctum eandem reddit

$$= \frac{\left(a + \frac{b}{n}\right)^3}{a^2}$$. Atque ita invenitur ſors in fine quarti momenti $= \frac{\left(a + \frac{b}{n}\right)^4}{a^3}$, ac tandem momento n:eſimo ſeu anni fine $\frac{\left(a + \frac{b}{n}\right)^n}{a^{n-1}}$, nimirum $a + b + \frac{b^2}{2a}$
$$+ \frac{b^3}{2.3a^2} + \frac{b^4}{2.3.4a^3} + \frac{b^5}{2.3.4.5a^4} + \frac{b^6}{2.3.4.5.6a^5} + \&c.,$$
ſive $a\left(1 + \frac{b}{a} + \frac{b^2}{2a^2} + \frac{b^3}{2.3a^3} + \frac{b^4}{2.3.4a^4} + \&c.\right)$.

Jam vero ex hyperbolae quadratura perſpectum eſt, quantitatem $1 + \frac{b}{a} + \frac{b^2}{2a^2} + \frac{b^3}{2.3a^3} + \&c.$ nihil aliud eſſe niſi numerum, cujus logarithmus hyperbolicus eſt $\frac{b}{a}$. Igitur pecunia finito anno Creditori debita aequatur facto ex priſtina ſorte a in numerum, cujus hyperbolicus logarithmus eſt ratio foenoris ad ſortem.

SCHOLION II.

8. In Bernoulliano Problemate ſumitur $\frac{b}{n}$ pro primi momenti foenore, propterea quod ex Problematis praeſcripto *pars* tempori *proportionalis uſurae annuae* accipienda eſt: at ſi in Problematibus ad compoſitum foenus, ſeu anatociſmum ſpectantibus capienda eſſet momentanea ſortis uſura primo inſtanti reſpondens, ea inveniretur $= \frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}} - a$; notum quippe eſt, in Problematibus hujuſmodi ſortem poſt datum quodlibet tempus t auctam exprimi ſemper per formulam $\frac{(a+b)^t}{a^{t-1}}$, ac proinde ſortem, primo elapſo momento $\frac{1}{n}$, evaſuram $\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}}$, ideoque inſtantaneam primi momenti uſuram fore $\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}} - a$. Explicata in ſeriem quantitate

$\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}}$ oritur infinitinomium $a + \frac{1}{n}b + \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{b^2}{2a}$

$$+\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\frac{b^3}{2.3a^2}+\frac{1}{n}\left(\frac{1}{n}-1\right)$$

$$\left(\frac{1}{n}-2\right)\left(\frac{1}{n}-3\right)\frac{b^4}{2.3.4a^3}+\frac{1}{n}\left(\frac{1}{n}-1\right)$$

$$\left(\frac{1}{n}-2\right)\left(\frac{1}{n}-3\right)\left(\frac{1}{n}-4\right)\frac{b^5}{2.3.4.5a^4}+\&c.$$

Porro in hoc infinitinomio factores $\frac{1}{n}-1$, $\frac{1}{n}-2$, $\frac{1}{n}-3$, &c. ob evaneſcentem ipſius $\frac{1}{n}$ valorem, degenerant in -1, -2, -3, $-$ &c. Igitur infinitinomium abit in $a+\frac{b}{n}-\frac{b^2}{2na}+\frac{b^3}{3na^2}-\frac{b^4}{4na^3}+\frac{b^5}{5na^4}-\&c.$

$$=a+\frac{1}{n}\left(b-\frac{b^2}{2a}+\frac{b^3}{3a^2}-\frac{b^4}{4a^3}+\frac{b^5}{5a^4}-\&c.\right).$$

Hinc conſequens eſt, inſtantaneum foenus $\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}}-a$ nanciſci formam $\frac{1}{n}\left(b-\frac{b^2}{2a}+\frac{b^3}{3a^2}-\frac{b^4}{4a^3}+\frac{b^5}{5a^4}-\frac{b^6}{6a^5}+\&c.\right)$, ſive etiam $\frac{1}{n}a\left(\frac{b}{a}-\frac{b^2}{2a^2}\right.$

$+\frac{b^3}{3a^3}-\frac{b^4}{4a^4}+\frac{b^5}{5a^5}-\&c.\Big)$ Eſt autem ex logarithmorum doctrina exploratum, quantitatem $\frac{b}{a}-\frac{b^2}{2a^2}+\frac{b^3}{3a^3}-\frac{b^4}{4a^4}+\&c.$ eſſe logarithmum hyperbolicum numeri $1+\frac{b}{a}$. Igitur accepto hic ſymbolo log. ad logarithmum non Briggianum, ſed hyperbolicum deſignandum reperitur uſura primo inſtanti reſpondens $=\frac{1}{n}\,a\,log.\left(1+\frac{b}{a}\right)$, vel $=a\,dt\,log.\left(1+\frac{b}{a}\right)$, ſi loco $\frac{1}{n}$ ſubſtituatur temporis t differentiale.

9. Si quis integrata expreſſione differentiali $a\,dt\,log.\left(1+\frac{b}{a}\right)$, inventoque integrali $at\,log.\left(1+\frac{b}{a}\right)$, contenderet, integrale iſtud aequari uſurae, quae reſpondet tempori indeterminato t, in fallaciam impingeret quo magis ſubtilem & latentem, eo ſollertius cavendam. Et ſane integrale quantitatis $a\,dt\,log.$ $\left(1+\frac{b}{a}\right)$ nihil aliud poteſt exprimere niſi ſummam omnium $a\,dt\,log.\left(1+\frac{b}{a}\right)$ in hypotheſi, quod ſingulis momentis foenus inſtantaneum ſit ſemper idem, nem-

pe $a\,dt\,log.\left(1+\frac{b}{a}\right)$, vel huic aequale $\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}} - a$; in qua certe hypotheſi uſura tempore indeterminato t parta evadit $at\,log.\left(1+\frac{b}{a}\right)$, & poſito $t = 1$ anno, fit illa $= a\,log.\left(1+\frac{b}{a}\right)$, quod etiam invenitur ducto foenore inſtantaneo conſtanti $\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}} - a$ in momentorum numerum n in anno contentorum, unde habetur $\frac{n(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}} - na$, hoc eſt $na\left(1+\frac{b}{na}+\frac{1}{n}\left(\frac{1}{n}-1\right)\frac{b^2}{2a^2}+\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\frac{b^3}{2.3a^3}\right.$ $\left.+\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\left(\frac{1}{n}-3\right)\frac{b^4}{2.3.4a^4}+\&c.\right)-na$, ſeu denique $a\left(\frac{b}{a}-\frac{b^2}{2a^2}+\frac{b^3}{3a^3}-\frac{b^4}{4a^4}+\frac{b^5}{5a^5}-\&c.\right)$, nimirum $a\,log.\left(1+\frac{b}{a}\right)$. Verum hypotheſis, quod inſtantaneum foenus in quaeſtionibus ad anatociſmum ſpectantibus conſtans & immutabile ſit ſingulis qui-

buſque

buſque momentis, falſa eſt & abſurda; quandoquidem ſingulis momentis foenus inſtantaneum variat, ac primo dumtaxat inſtante formam accipit $adt\ log.\left(1+\frac{b}{a}\right)$, caeteris autem omnibus aliam ſemper atque aliam. Propterea quantitas $adt\ log.\left(1+\frac{b}{a}\right)$ haudquaquam ſpectari poteſt veluti elementum foenoris una cum tempore t fluentis, neque proinde ejus integrale haberi poteſt pro foenore, quod indeterminato tempore t producatur.

10. Verum ac proprie dictum uſurae una cum tempore t fluentis elementum invenitur hoc pacto: Exploratum eſt, ſortem a tempore t evadere $a\left(1+\frac{b}{a}\right)^t$, & tempore $t+dt$, ſeu momento ſubſequenti fieri $a\left(1+\frac{b}{a}\right)^{t+dt}$; igitur differentia inter ſortes duas infinite propinquas, nimirum $a\left(1+\frac{b}{a}\right)^{t+dt}$ $-a\left(1+\frac{b}{a}\right)^t$ repraeſentabit inſtantaneum foenus partum tempuſculo infiniteſimo dt poſt tempus quodvis indeterminatum t, quod equidem conſtituit verum ac proprie dictum foenoris elementum. Idemque reperi-

tur ope hujus analogiae, quemadmodum ſe habet ſors a ad foenus ſuum inſtantaneum momenti primi $a\left(1+\frac{b}{a}\right)^{dt}-a$, ita ſe habet ſors $a\left(1+\frac{b}{a}\right)^{t}$ poſt tempus indeterminatum t ad foenus ſuum inſtantaneum $a\left(1+\frac{b}{a}\right)^{t+dt}-a\left(1+\frac{b}{a}\right)^{t}$.

11. Nunc autem ſuſcepta integratione elementi $a\left(1+\frac{b}{a}\right)^{t+dt}-a\left(1+\frac{b}{a}\right)^{t}$ invenietur accurate foenus tempore quolibet indeterminato t productum. Etenim quantitas $a\left(1+\frac{b}{a}\right)^{t+dt}-a\left(1+\frac{b}{a}\right)^{t}$, ſeu $a\left(1+\frac{b}{a}\right)^{t}\left(1+\frac{b}{a}\right)^{dt}-a\left(1+\frac{b}{a}\right)^{t}$, explicato binomio $\left(1+\frac{b}{a}\right)^{dt}$, deprehenditur $=a\left(1+\frac{b}{a}\right)^{t}$
$$\left(1+\frac{bdt}{a}+dt(dt-1)\frac{b^2}{2a^2}+dt(dt-1)(dt-2)\frac{b^3}{2.3a^3}\right.$$
$$\left.+dt(dt-1)(dt-2)(dt-3)\frac{b^4}{2.3.4a^4}+\&c.\right)$$
$-a\left(1+\frac{b}{a}\right)^{t}$. Porro in hac expreſſione factores $dt-1$, $dt-2$, $dt-3$, &c. evaneſcente dt mutantur in -1, -2, -3, &c. & proinde ipſa expreſ-

ſic degenerat in $a\left(1+\frac{b}{a}\right)^t \left(1+\frac{bdt}{a}-\frac{b^2dt}{2a^2}+\frac{b^3dt}{3a^3}-\frac{b^4dt}{4a^4}+\frac{b^5dt}{5a^5}-\&c.\right)-a\left(1+\frac{b}{a}\right)^t$, hoc eſt in $a\left(1+\frac{b}{a}\right)^t dt\left(\frac{b}{a}-\frac{b^2}{2a^2}+\frac{b^3}{3a^3}-\frac{b^4}{4a^4}+\frac{b^5}{5a^5}-\frac{b^6}{6a^6}+\&c.\right)$, ſeu tandem in $a\left(1+\frac{b}{a}\right)^t dt \log. \left(1+\frac{b}{a}\right)$. Conſtat autem ex quantitatum exponentialium integratione, eſſe

$$\int a\left(1+\frac{b}{a}\right)^t dt \log. \left(1+\frac{b}{a}\right)=a\left(1+\frac{b}{a}\right)^t$$

$+$ conſt., & quum foenus temporis initio, ſeu evaneſcente t nullum ſit, fiet conſt. $=-a$; proindeque integrale completum $=a\left(1+\frac{b}{a}\right)^t-a$, quod uſuram praebet tempori cuilibet indeterminato reſpondentem, & (poſito $t=1$ anno) uſuram dat b, quemadmodum oportet.

12. Caeterum quod perſpicue demonſtrat, a formula $\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}}-a$ rite repraeſentari inſtantaneum foenus primo momento caducum, eſt ſequens ratioci-

nati[illegible], cujus ope ab ipſo primi momenti foenore aſcendere licet gradatim uſque ad foenus annuum b. Sane, ſi ſors a primo momento gignit uſuram $\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}} - a$, ſors $a + \frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}} - a$, ſeu $\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}}$ pariet ſecundo momento uſuram $\frac{(a+b)^{\frac{2}{n}}}{a^{\frac{2}{n}-1}} - \frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}}$, qua addita uſurae primi momenti oritur $\frac{(a+b)^{\frac{2}{n}}}{a^{\frac{2}{n}-1}} - a$ pro uſura duobus primis momentis producta. Rurſus ſi ſors a momento primo gignit foenus $\frac{(a+b)^{\frac{1}{n}}}{a^{\frac{1}{n}-1}} - a$, ſors $\frac{(a+b)^{\frac{2}{n}}}{a^{\frac{2}{n}-1}}$ gignet momento tertio uſuram $\frac{(a+b)^{\frac{3}{n}}}{a^{\frac{3}{n}-1}} - \frac{(a+b)^{\frac{2}{n}}}{a^{\frac{2}{n}-1}}$; hacque addita uſurae duorum praece-

dentium momentorum prodit uſura trium priorum momentorum $\frac{(a+b)^{\frac{3}{n}}}{a^{\frac{3}{n}-1}} - a$. Atque ita invenitur quatuor priorum momentorum uſura $= \frac{(a+b)^{\frac{4}{n}}}{a^{\frac{4}{n}-1}} - a$; ac denique momentorum omnium n annum integrum conſtituentium uſura $= \frac{(a+b)^{\frac{n}{n}}}{a^{\frac{n}{n}-1}} - a = a + b - a = b$, qualem eſſe oportet.

13. Neque aliter ex formula altera aequipollenti $\frac{1}{n} a \, log. \left(1 + \frac{b}{a}\right)$ rite tractata reperitur. Et revera ſi a ſorte a provenit primo inſtanti uſura $\frac{1}{n} a \, log. \left(1 + \frac{b}{a}\right)$, a ſorte $a + \frac{1}{n} a \, log. \left(1 + \frac{b}{a}\right)$ proveniet ſecundo inſtanti foenus $\frac{1}{n} a \, log. \left(1 + \frac{b}{a}\right) + \frac{1}{n^2} a \left(log. (1 + \frac{b}{a})\right)^2$, quod ſi addatur foenori primi inſtantis, oritur foenus duorum inſtantium primorum $\frac{2}{n} a \, log. \left(1 + \frac{b}{a}\right)$

$+\frac{1}{n}a\left(\log.\left(1+\frac{b}{a}\right)\right)^2 = a\left(1+\frac{1}{n}\log.\left(1+\frac{b}{a}\right)\right)^2$ $-a$. Rurſus ſi ſors a progignit primo inſtanti uſuram $\frac{1}{n}a\log.\left(1+\frac{b}{a}\right)$, ſors $a\left(1+\frac{1}{n}\log.\left(1+\frac{b}{a}\right)\right)^2$ duobus primis inſtantibus cumulata generat tertio inſtanti uſuram $a\left(1+\frac{1}{n}\log.\left(1+\frac{b}{a}\right)\right)^2 \times \frac{1}{n}\log.\left(1+\frac{b}{a}\right)$. Hac collecta in unam ſummam cum uſura duorum inſtantium praecedentium, prodit uſura trium priorum inſtantium $= a\left(1+\frac{1}{n}\log.\left(1+\frac{b}{a}\right)\right)^3 - a$. Eodemque modo invenitur uſura quatuor priorum inſtantium $= a\left(1+\frac{1}{n}\log.\left(1+\frac{b}{a}\right)\right)^4 - a$, ac demum uſura inſtantibus omnibus n progenita

$$= a\left(1+\frac{1}{n}\log.\left(1+\frac{b}{a}\right)\right)^n - a.$$

Explicata porro in ſeriem poteſtate $\left(1+\frac{1}{n}\log.\left(1+\frac{b}{a}\right)\right)^n$, & in termini cujuſvis coëfficiente contemptis numeris prae infinito n evaneſcentibus, oritur ipſa poteſtas

$$= 1+\log.\left(1+\frac{b}{a}\right)+\frac{1}{2}\left(\log.\left(1+\frac{b}{a}\right)\right)^2$$

$+\frac{1}{2.3}\left(\log.\left(1+\frac{b}{a}\right)\right)^3+\frac{1}{2.3.4}\left(\log.\left(1+\frac{b}{a}\right)\right)^4+\&c.,$

quae expreſſio ex logarithmorum hyperbolicorum doctrina aequatur numero, cujus hyperbolicus logarithmus eſt ipſe ſecundus ſeriei terminus $\log.\left(1+\frac{b}{a}\right)$; proindeque fit ſeries ipſa $=1+\frac{b}{a}$. Quare uſura primo anno parta $a\left(1+\frac{1}{n}\log.\left(1+\frac{b}{a}\right)\right)^n-a$ invenitur $=a\left(1+\frac{b}{a}\right)-a=b$, ſicuti expectabatur.

14. Haud inutile erit animadvertere, differentiale foenoris tempore quovis cumulati idem penitus eſſe ac differentiale ſortis eidem tempori reſpondentis. Nam ſors poſt datum quodlibet tempus t fit, uti palam eſt, $a\left(1+\frac{b}{a}\right)^t$, & poſt tempus proxime ſequens $t+dt$ ea evadit $a\left(1+\frac{b}{a}\right)^{t+dt}$: quamobrem ejus differentiale erit differentia ſortis ipſius in duobus ſtatibus infinite proximis conſideratae, hoc eſt fiet ejus differentiale $=a\left(1+\frac{b}{a}\right)^{t+dt}-a\left(1+\frac{b}{a}\right)^t$

$= a\left(1+\frac{b}{a}\right)^{t} dt \, log.\left(1+\frac{b}{a}\right)$, quemadmodum foenoris. Neque id mirum videri debet, quum expreſſio ſortis $a\left(1+\frac{b}{a}\right)^{t}$ non diſcrepet ab expreſſione foenoris $a\left(1+\frac{b}{a}\right)^{t} - a$ niſi conſtanti quantitate a, exploratumque alias ſit binarum variabilium conſtanti quantitate inter ſe differentium eamdem eſſe fluxionem, ſeu idem elementum aut differentiale.



DISQUISITIO XV.

DE INSIGNIBUS QUIBUSDAM MOTUS VERTICALIS PROPRIETATIBUS IN CORPORIBUS ASCENDENTIBUS ET LIBERE DESCENDENTIBUS.

1. INter Auctores quamplurimos, qui de Corporum Motibus magna ingenii dexteritate diſſeruerunt, atque hanc utiliorem praeſtantioremque Mechanicae partem ſcriptis editis illuſtrarunt, nemo, quod ſciam, data opera inquiſivit in ſingulares quaſdam ac prope admirandas Motus Verticalis affectiones in illis corporibus, quae verticaliter ſurſum dato impetu projiciuntur libereque deorſum labuntur, in eaque vis centralis, ſeu gravitatis terreſtris hypotheſi, quam non imaginatio ſibi fingit, ſed natura ipſa poſtulat, experimenta confirmant, Coelumque ipſum, ſi ita loqui fas eſt, Terra, ac NEWTONUS demonſtrant. Hanc quidem Naturae hypotheſim de vi centrali quadratis diſtantiarum a centro reciproce proportionali excoluit in elegantiſſimo Opuſculo *De Corporum Motu Rectilineo, & Curvilineo*

in Mediis non reſiſtentibus §. 48. Coeleſtinus ROLLIUS, praecipua rectilinei motus ſymptomata in hac aſſumptione virium evolvens; ſed admirabiliores affectiones ſilentio praeteriit, motumque dumtaxat corporum ſponte deſcendentium inveſtigavit, & analyſi denique vix delibata longiorem moleſtioremque syntheſis rationem perſequi maluit. Conſtitui itaque ſingularia iſthaec Verticalis Motus symptomata ſcrutari, eaque per breviorem ſeverioremque Analyſeos viam diligenter evolvere atque explicare.

2. Projiciatur grave verticaliter ſurſum dato impetu vel celeritate c in hypotheſi gravitatis terreſtris decreſcentis in ratione duplicata inverſa diſtantiarum a Terrae centro, in vera nimirum Naturae hypotheſi, qualem terreſtria coeleſtiaque phoenomena certo ſuadent & apprime demonſtrant. Sit Terrae ſemidiameter r; altitudo, ad quam grave pervenit tempore t, velocitate v, ſit x, gravitas demum acceleratrix in terreſtri ſuperficie dicatur g. Jamvero, quoniam eſt ex hypotheſi vis acceleratrix uti reciproce quadratum diſtantiae a Terrae centro, erit icciroo $(r+x)^2 : r^2 : :$ $g : \frac{gr^2}{(r+x)^2}$ = vi acceleratrici gravitatis in altitudine x ſupra Telluris ſuperficiem. Notum porro eſt ex Mechanicis, vim ductam in tempuſculum aequari extin-

ᶜto elemento velocitatis; propterea fiet $\frac{gr^2 dt}{(r+x)^2} = -dv$.

Quocirca ob $dt = \frac{dx}{v}$, obtinebitur $\frac{gr^2 dx}{(r+x)^2} = -vdv$; atque adeo $\int \frac{gr^2 dx}{(r+x)^2} = \int -vdv$, ſeu $\frac{gr^2}{r+x} = \frac{1}{2}v^2 +$ Conſt. Invenitur autem Conſt. $= gr - \frac{1}{2}c^2$ eo quod initio motus exiſtente $x = 0$, evadit $v = c$. Igitur nanciſcimur $\frac{1}{2}v^2 = \frac{gr^2}{r+x} - gr + \frac{1}{2}c^2$, ſeu $v = \surd\left(c^2 - \frac{2grx}{r+x}\right)$.

3. Evidens jam eſt ex hac aequatione $v = \surd\left(c^2 - \frac{2grx}{r+x}\right)$, in corporibus ſurſum projectis aliquam ſemper ſupereſſe velocitatem reſiduam v poſt emenſam altitudinem x quotieſcumque fuerit $c^2 > \frac{2grx}{r+x}$, ſive $x < \frac{c^2 r}{2gr - c^2}$; ita ut ſi initialis velocitas c deſignet ſpatium motu aequabili deſcribendum a corpore unius ſecundi tempore, & vis gravitatis acceleratrix g in ſuperficie terreſtri ſpatium deſignet, quod unius pariter ſecundi tempore corpus percurreret uniformi motu ex ipſius gravitatis g actione, ſpatium videlicet pe-

dum pariſienſium 30, 2, remanere debeat in corpore aſcendente velocitas aliqua v quoties habeatur $c^2 > \frac{60,4rx}{r+x}$, ſive $x < \frac{c^2 r}{60,4r+c^2}$, hoc eſt (ſumpta ſemidiametro terreſtri r pedum pariſienſ. 19741200) $c^2 > \frac{1192368480x}{19741200+x}$ vel $x > \frac{19741200\,c^2}{1192368480-c^2}$. Tota vero aſcendentis corporis extinguetur velocitas impreſſa ubi fuerit $c^2 = \frac{2grx}{r+x} = \frac{1192368480\,x}{19741200+x}$, vel $x = \frac{c^2 r}{2gr-c^2} = \frac{19741200\,c^2}{1192368480-c^2}$.

4. Reſidua corporis ſurſum projecti celeritas v invenitur imaginaria aut abſurda quotieſcumque $c^2 > \frac{2grx}{r+x}$, vel $x > \frac{c^2 r}{2gr-c^2}$; quod ſane indicat, altitudinem x acceptam fuiſſe juſto majorem, ad quam nimirum projectum non pertingit. Capta vero altitudine x infinita, reperitur reſidua velocitas v ſemper repugnans quoties fuerit $c^2 < 2gr. < 1192368480$ ped.; ſed ſumpta c^2 vel $=$, vel $>$ 1192368480 ped., ſive $c =$, aut $>$ 34530, 69 ped., velocitas reſidua v invenitur realis, videlicet aut nulla aut poſitiva, infinita licet exiſtente altitudine, ad quam corpus debet eniti.

5. En igitur Theorema prorſus mirificum & ſin-

gulare: *Datur velocitas finita, eaque arctis limitibus circumſcripta, & ipſa celeritate luminis immaniter minor, quacum ſi grave corpus ſurſum projiciatur, ſublata medii reſiſtentia, veraque admiſſa gravitatis decreſcentis hypotheſi in ratione duplicata inverſa diſtantiarum a Terrae centro, idem perveniret ad altitudinem infinitam, ſeu qualibet data & aſſignabili majorem.* Haec autem velocitas talis eſt, ut corpus ab ipſa animatum pedes tantum 34531 ſingulis ſecundis aequabiliter conficeret. Sane in formula $v = \sqrt{\left(c^2 - \frac{2grx}{r+x}\right)}$, fit $v = 0$ quando $c = \sqrt{\frac{2grx}{r+x}}$, hoc eſt (facta $x = \infty$) quando $c = \sqrt{2gr} = 34531$. Hinc proponi poteſt hoc elegantiſſimum Problema: *Invenire velocitatem projectionis, quae in gravi verticaliter aſcendente non pereat niſi poſt emenſum ſpatium infinitum in vera hypotheſi gravitatis decreſcentis* &c. Reperietur hujuſmodi celeritas qualis modo definita fuit, nimirum nec immenſa nec ultra modum ingens.

6. Sed proprietatem adeo inſignem & inexpecttatam juvat aliter demonſtrare: Labatur grave corpus ab altitudine ſemidiametrorum terreſtrium n facta computatione ab ipſo terrae centro; eritque $\frac{gr^2\,dx}{(nr - x)^2}$

$= \int udu$, dicta x altitudine deſcendendo percurſa, u velocitate illo deſcenſu acquiſita. Quare $\int \frac{gr^2 dx}{(nr - x)^2} = \int udu$, ſeu $\frac{gr^2}{nr - x} = \frac{1}{2} u^2 +$ Conſt. $= \frac{1}{2} u^2 + \frac{gr}{n}$. Igitur in fine lapſus, quando nempe corpus telluris ſuperficiem attigit, ubi $x = (n - 1) r$, evadit $u^2 = 2gr - \frac{2gr}{n}$. Sit modo $n = \infty$, cadat ſcilicet corpus ab infinita altitudine, ſeu ſemidiametris terreſtribus numero infinitis aequali, & orietur $u^2 = 2gr$, ſive $u = \sqrt{2gr} = 34531$ ped. Quapropter corpus ex infinita lapſum altitudine, & telluris ſuperficiem jam attingens acquirit velocitatem dumtaxat finitam, videlicet velocitatem pedum tantummodo 34531 ſingulis ſecundis. Igitur e contrario corpus hac ipſa finita velocitate ſurſum emiſſum ad altitudinem pervenit infinitam.

7. Alia inſignis motuum iſtiuſmodi proprietas ita ſe habet: *Si initialis velocitas* c, *corpori ſurſum projecto impreſſa, infinita fuerit, ea poſt emenſam a corpore altitudinem infinitam non modo non extinguitur, vel convertitur in velocitatem finitam, ſed adhuc perſeverat infinita, nec ab ipſa initiali diverſa*. Id liquet ex ſuperiori formula $v = \sqrt{\left(c^2 - \frac{2grx}{r+x}\right)}$, quae aſſumptis x, & c in-

finitis mutatur in $\sqrt{(c^2 - 2gr)} = c - \frac{gr}{c} - \frac{g^2 r^2}{2c^3} - \frac{g^3 r^3}{2c^5} - \frac{5 g^4 r^4}{8 c^7}$ — &c.; quae quidem ſeries non differt a primo termino c in hac hypotheſi ipſius $c = \infty$.

8. Pergo modo, & quaero tempus t per functionem velocitatis v repraeſentatum, in corporibus aſcendentibus. Ex formula $v = \sqrt{\left(c^2 - \frac{2grx}{r+x}\right)}$ inſero $x = \frac{c^2 r - rv^2}{2gr - c^2 + v^2}$, & facta differentiatione elicio $dx = -\frac{4gr^2\,vdv}{(2gr - c^2 + v^2)^2}$; atque inde $dt = \frac{dx}{v} = -\frac{4gr^2\,dv}{(2gr - c^2 + v^2)^2}$. Jamvero hujus quantitatis integrale ita exquiro: Pono $\int -\frac{4gr^2\,dv}{(2gr - c^2 + v^2)^2} = \frac{Av + B}{2gr - c^2 + v^2} - \int \frac{4gr^2\,D\,dv}{2gr - c^2 + v^2}$, ac ſumo differentialia, habeoque $-\frac{4gr^2\,dv}{(2gr - c^2 + v^2)^2} = \frac{(2gr - c^2)\,Adv - Av^2 dv - 2B\,vdv}{(2gr - c^2 + v^2)^2} - \frac{4gr^2\,D\,dv}{2gr - c^2 + v^2}$. Hinc reductione terminorum facta ad eundem Denominatorem conſequor hanc aequationem

$$\left.\begin{array}{lll} Av^2 & -2Bv & +2grA \\ -4gr^2Dv^2 & & -c^2A \\ & & -8g^2r^3D \\ & & +4gr^2c^2D \\ & & +4gr^2 \end{array}\right\} = 0$$

Ex hac hac vero ſequentes nanciſcor aequalitates $A + 4gr^2D = 0$, $2B = 0$, $2grA - c^2A - 8g^2r^3D + 4gr^2c^2D + 4gr^2 = 0$ Propterea fit $A = -\frac{2gr^2}{2gr - c^2}$, $B = 0$, $D = \frac{1}{4gr - 2c^2}$. Igitur

$$S - \frac{4gr^2\,dv}{(2gr - c^2 + v^2)^2} = -\frac{2gr^2 v}{(2gr - c^2)(2gr - c^2 + v^2)} - S\frac{2gr^2\,dv}{(2gr - c^2)(2gr - c^2 + v^2)}.$$

Liquet porro, expreſſionem $S\frac{2gr^2\,dv}{(2gr - c^2)(2gr - c^2 + v^2)}$ in hypotheſi $2gr > c^2$ aequalem eſſe quantitati $\frac{2gr^2}{(2gr - c^2)^2}$ ductae in arcum circuli, cujus radius eſt $\sqrt{(2gr - c^2)}$, tangens $= v$, vel etiam aequalem quantitati $\frac{2gr^2}{(2gr - c^2)^{\frac{3}{2}}}$ multiplicatae per arcum circuli, cujus ſemidiameter eſt $= 1$, tangens $= \frac{v}{\sqrt{(2gr - c^2)}}$. Habebitur itaque

$$S - \frac{4gr^2\, dv}{(2gr - c^2 + v^2)^2} = - \frac{2gr^2 v}{(2gr - c^2)(2gr - c^2 + v^2)}$$

$$- \frac{2gr^2}{(2gr - c^2)^{\frac{3}{2}}} \times Arc.\ tang.\ \frac{v}{\sqrt{(2gr - c^2)}}$$

$+$ Conſt. $= t$. Quum autem motus initio, ſeu aſſumpto $t = 0$ evadat $v = c$, invenitur Conſt.

$$= \frac{rc}{2gr - c^2} + \frac{2gr^2}{(2gr - c^2)^{\frac{3}{2}}} \times Arc.\, tang.\, \frac{c}{\sqrt{(2gr - c^2)}}.$$

Quare poſtremo nanciſcimur $t = \frac{rc}{2gr - c^2} -$

$$\frac{2gr^2 v}{(2gr - c^2)(2gr - c^2 + v^2)} + \frac{2gr^2}{(2gr - c^2)^{\frac{3}{2}}}$$

$$\left(Arc.\ tang.\ \frac{c}{\sqrt{(2gr - c^2)}} - Arc.\, tang.\, \frac{v}{\sqrt{(2gr - c^2)}}\right);$$

& quoniam datis duobus arcubus φ, λ eſt *tang.* $(\varphi - \lambda) = \frac{tang.\, \varphi - tang.\, \lambda}{1 + tang.\, \varphi\ tang.\, \lambda}$, erit denique t

$$= \frac{rc}{2gr - c^2} - \frac{2gr^2 v}{(2gr - c^2)(2gr - c^2 + v^2)} + \frac{2gr^2}{(2gr - c^2)^{\frac{3}{2}}} \times$$

$$Arc.\ tang.\ \frac{(c - v)\sqrt{(2gr - c^2)}}{2gr - c^2 + cv}.$$

9. Si jam velocitas initialis c, aſcendenti corpori communicata, fuerit $= \sqrt{2gr} = 34531$ ped. unius ſecundi tempore, facile oſtendimus, velocitatem ipſam

non perire niſi poſt elapſum tempus infinitum. In hac enim hypotheſi aequatio differentialis $dt = -\frac{4gr^2\,dv}{(2gr - c^2 + v^2)^2}$ mutatur in $-\frac{4gr^2\,dv}{v^4}$; atque hinc eruitur $t = \frac{4gr^2}{3v^3} - \frac{4gr^2}{3c^3}$. Proinde ſumpto $v = 0$ oritur $t = \infty$. Quapropter *ſi corpori ſurſum projecto velocitas imprimatur finita, nec admodum ingens, pedum nempe* 34531. *ſingulis ſecundis, ea non extinguitur niſi poſt infinitum tempus, & corpus pergit ſemper aſcendere contra gravitatis directionem, quin unquam ad quietem redigatur.*

10. Fingamus nunc, velocitatem initialem c majorem quam $\sqrt{2gr}$; inveniemus, expreſſionem temporis t §. 8. imaginariam eſſe; quod quidem nos docet, formulae $\int -\frac{4gr^2\,dv}{(2gr - c^2 + v^2)^2}$ integrale aliunde quam a circuli rectificatione pendere: & revera exploratum alias eſt, differentialem formulam $-\frac{4gr^2\,dv}{(2gr - c^2 + v^2)^2}$ in hypotheſi $c^2 > 2gr$ per hyperbolae quadraturam integrari. Itaque fiat $c^2 - 2gr = h^2$, eritque $dt = -\frac{4gr^2\,dv}{(v^2 - h^2)^2} = -\frac{4gr^2\,dv}{(v+h)^2(v-h)^2}$.

Capio igitur $-\frac{4gr^2\,dv}{(v^2-h^2)^2} = \frac{Avdv + Bdv}{(v+h)^2}$ $+\frac{Cvdv + Ddv}{(v-h)^2}$; terminos redigo ad eundem denominatorem, & aequationem nanciſcor

$$\begin{array}{llll} Av^3 - 2hAv^2 & + h^2Av & + h^2B & \\ + Cv^3 + \quad Bv^2 & - 2hBv & + h^2D & = 0. \\ \quad + 2hCv^2 & + h^2Cv & + 4gr^2 & \\ + \quad Dv^2 & + 2hDv & & \end{array}$$

Ex hac vero ſequentes deducuntur aequalitates

$$A + C \ldots\ldots\ldots\ldots = 0;$$

$$-2hA + B + 2hC + D = 0;$$

$$h^2A - 2hB + h^2C + 2hD = 0;$$

$$h^2B + h^2D + 4gr^2 \ldots = 0;$$

atque inde infertur

$$A \ldots\ldots = -\frac{gr^2}{h^3};$$

$$B \ldots\ldots = -\frac{2gr^2}{h^2};$$

$$C \ldots\ldots = \frac{gr^2}{h^3};$$

$$D \ldots\ldots = -\frac{2gr^2}{h^2}.$$

Quamobrem $-\frac{4gr^2 dv}{(v^2 - h^2)^2} = -\frac{gr^2 vdv}{h^3(v+h)^2}$ $+\frac{gr^2 vdv}{h^3(v-h)^2} - \frac{2gr^2 dv}{h^2(v+h)^2} - \frac{2gr^2 dv}{h^2(v-h)^2}$.

Nunc autem protinus invenitur $\int -\frac{gr^2 vdv}{h^3(v+h)^2} = -\frac{gr^2}{h^2(v+h)} - \frac{gr^2}{h^3} \times$ $log.(v+h)$; $\int \frac{gr^2 vdv}{h^3(v-h)^2} = -\frac{gr^2}{h^2(v-h)} + \frac{gr^2}{h^3} \times$ $log.(v-h)$; $\int -\frac{2gr^2 dv}{h^2(v+h)^2} = \frac{2gr^2}{h^2(v+h)}$; $\int -\frac{2gr^2 dv}{h^2(v-h)^2} = \frac{2gr^2}{h^2(v-h)}$. Igitur t $= \int -\frac{4gr^2 dv}{(v^2 - h^2)^2} = \frac{gr^2}{h^2(v+h)} + \frac{gr^2}{h^2(v-h)}$ $+ \frac{gr^2}{h^3} log. \frac{v-h}{v+h}$ + Conſt. $= \frac{2gr^2 v}{h^2(v^2 - h^2)} + \frac{gr^2}{h^3} \times$ $log. \frac{v-h}{v+h}$ + Conſt. Porro initio motus, ſeu evaneſcente t, abit v in c; proinde fit Conſt. $= -\frac{rc}{h^2}$ $- \frac{gr^2}{h^3} log. \frac{c-h}{c+h}$; adeoque tandem $t = \frac{2gr^2 v}{h^2(v^2 - h^2)}$

$$-\frac{rc}{h^2} + \frac{gr^2}{h^3}\log.\frac{(v-h)\,(c+h)}{(v+h)\,(c-h)}.$$

11. Inventa expreſſione temporis per velocitatem, reſtat nunc ut tempus ipſum per ſpatii functionem repraeſentatum detegamus. Itaque formula §. 2. $v = \sqrt{\left(c^2 - \frac{2grx}{r+x}\right)}$ mutetur primo in $v = \sqrt{\frac{c^2x - 2grx + c^2r}{r+x}}$;

tum ob $dt = \frac{dx}{v}$ obveniet $dt = dx\sqrt{\frac{r+x}{c^2x - 2grx + rc^2}}$

$= dx\sqrt{\frac{r+x}{h^2x + rc^2}}$. Ut integratio abſolvatur, fiat

$\sqrt{\frac{r+x}{h^2x + rc^2}} = z$, eritque $dx = \frac{(2rc^2 - 2rh^2)\,z\,dz}{(h^2z^2 - 1)^2}$,

adeoque $dt = \frac{(2rc^2 - 2rh^2)\,z^2\,dz}{(h^2z^2 - 1)^2} = \frac{2rc^2 - 2rh^2}{h^4} \times$

$$\frac{z^2\,dz}{\left(z^2 - \frac{1}{h^2}\right)^2} = \frac{2rc^2 - 2rh^2}{h^4} \times \frac{z^2\,dz}{\left(z - \frac{1}{h}\right)^2\left(z + \frac{1}{h}\right)^2}.$$

Ut hujus quantitatis integrale eliciatur, pono

$$\frac{z^2\,dz}{\left(z - \frac{1}{h}\right)^2\left(z + \frac{1}{h}\right)^2} = \frac{Az\,dz + B\,dz}{\left(z + \frac{1}{h}\right)^2} + \frac{Cz\,dz + D\,dz}{\left(z - \frac{1}{h}\right)^2},$$

revocatiſque fractionibus ad eundem denominatorem aequationem ſequentem conſequor

$$\left.\begin{aligned}
&Az^3 - \frac{2Az^2}{h} + \frac{Az}{h^2} + \frac{B}{h^2} \\
&+ Cz^3 + Bz^2 - \frac{2Bz}{h} + \frac{D}{h^2} \\
&+ \frac{2Cz^2}{h} + \frac{Cz}{h^2} \\
&+ Dz^2 + \frac{2Dz}{h} \\
&- z^2
\end{aligned}\right\} = 0.$$

Ex hac porro deducuntur aequalitates

$$A + C \ldots\ldots\ldots\ldots = 0;$$

$$-\frac{2A}{h} + B + \frac{2C}{h} + D - 1 = 0;$$

$$\frac{A}{h^2} - \frac{2B}{h} + \frac{C}{h^2} + \frac{2D}{h} \ldots = 0;$$

$$\frac{B}{h^2} + \frac{D}{h^2} \ldots\ldots\ldots\ldots = 0:$$

Hinc invenitur

$$A \ldots\ldots = -\tfrac{1}{4}h;$$

$$B \ldots\ldots = 0;$$

$$C \ldots\ldots = \tfrac{1}{4}h;$$

$$D \ldots\ldots = 0.$$

Igitur

$$dt = \frac{(2rc^2 - 2h^2 r) z^2 dz}{h^4 (z + \frac{1}{h})^2 (z - \frac{1}{h})^2} = - \frac{(rc^2 - rh^2) z dz}{2h^3 (z + \frac{1}{h})^2}$$

$$+ \frac{(rc^2 - rh^2) z dz}{2h^3 (z - \frac{1}{h})^2} = - \frac{gr^2 z dz}{h^3 (z + \frac{1}{h})^2} + \frac{gr^2 z dz}{h^3 (z - \frac{1}{h})^2}.$$

Jam vero eſt $\int - \frac{gr^2 z dz}{h^3 (z + \frac{1}{h})^2} = - \frac{gr^2}{h^4 (z + \frac{1}{h})}$ $- \frac{gr^2}{h^3} log. (z + \frac{1}{h})$; & $\int \frac{gr^2 z dz}{h^3 (z - \frac{1}{h})^2} = - \frac{gr^2}{h^4 (z - \frac{1}{h})}$ $+ \frac{gr^2}{h^3} log. (z - \frac{1}{h})$: iccirco erit $t = - \frac{2 gr^2 z}{h^4 (z^2 - \frac{1}{h^2})}$ $+ \frac{gr^2}{h^3} log. \frac{z - \frac{1}{h}}{z + \frac{1}{h}} +$ Conſt. $= \frac{1}{h^2} \sqrt{(r + x)}$ $(h^2 x + rc^2) + \frac{gr^2}{h^3} log. \frac{\sqrt{(r+x)} - \frac{1}{h} \sqrt{(h^2 x + rc^2)}}{\sqrt{(r+x)} + \frac{1}{h} \sqrt{(h^2 x + rc^2)}}$ + Conſt. Quum autem x evaneſcat una cum t, oritur inde Conſt. $= - \frac{rc}{h^2} - \frac{gr^2}{h^3} log. \frac{h - c}{h + c}$; ac demum $t = \frac{1}{h^2} \sqrt{(r + x)} (h^2 x + rc^2) - \frac{rc}{h^2} + \frac{gr^2}{h^3} \times$

$$log.\frac{[h\sqrt{(r+x)}-\sqrt{(h^2x+rc^2)}](h+c)}{[h\sqrt{(r+x)}+\sqrt{(h^2x+rc^2)}](h-c)}.$$

12. Expendamus nunc paullo diligentius formulam ſuperiorem §. 10., expreſſionem videlicet temporis per velocitatem, hoc eſt $t = \frac{2gr^2v}{h^2(v^2-h^2)} - \frac{rc}{h^2} + \frac{gr^2}{h^3}\,log.\,\frac{(v-h)(c+h)}{(v+h)(c-h)}$, atque in anteceſſum animo reputemus, verſari nos jam in hypotheſi celeritatis initialis $c > \sqrt{2gr}$. Ponamus itaque, exceſſum ipſius c^2 ſupra $2gr$ eſſe $= \varphi^2$: hinc prodibit valor temporis $t = \frac{2gr^2v}{\varphi^2(v^2-\varphi^2)} - \frac{rc}{\varphi^2} + \frac{gr^2}{\varphi^3} \times$ $log.\frac{(v-\varphi)(c+\varphi)}{(v+\varphi)(c-\varphi)}$. Ex hoc autem temporis valore palam fit, capto $v > \varphi$ velocitatem aſcendenti corpori impreſſam decreſcere uſque ad valorem utcumque paullo majorem quam φ tempore ſemper finito ac determinato. At vero ſi accipiatur $v = \varphi$, invenitur $t = \frac{2gr^2\varphi}{0} + \frac{gr^2}{\varphi^3}\,log.\,0 = \infty$; conſtat enim $log.\,0$ infinitum eſſe negativum, ſed in immenſum minus

quam infinitum alterum $\frac{2gr^2\varphi}{0}$. En itaque mira atque inſignis Motuum iſtiuſmodi affectio: *Si corpori ſurſum emiſſo velocitas imprimatur major quam* $\sqrt{2gr}$, *ſeu major quam* 34531. *ped. pro ſingulis ſecundis*, *exceſſu exiſtente quolibet* k, *ea non decreſcet ad exceſſum uſque* k *niſi poſt tempus infinitum elapſum*.

13. Valor alter temporis t per ſpatii functionem expreſſus §. 11. praebet in hac hypotheſi, qua ſumitur $c^2 = 2gr + \varphi^2$, aequationem $t = \frac{1}{\varphi^2}\times$

$$\sqrt{(r+x)(\varphi^2 x + rc^2)} - \frac{rc}{\varphi^2} + \frac{2gr^2}{\varphi^3}\times$$

$$log.\frac{[\varphi\sqrt{(r+x)} - \sqrt{(\varphi^2 x + rc^2)}](\varphi + c)}{[\varphi\sqrt{(r+x)} + \sqrt{(\varphi^2 x + rc^2)}](\varphi - c)}.$$ Hinc ergo aſſumpta altitudine x infinita reperitur $t = \frac{x}{\varphi} + \frac{2gr^2}{\varphi^3}\, log.\, 0$; eſt autem $log.\, 0$ infinitum negativum, ſed in immenſum minus quam infinitum $\frac{x}{\varphi}$, quemadmodum ex Infinitorum Geometria facile oſtenditur. Propterea elicitur $t = \infty$.

14. Si denique tum celeritas initialis c, tum altitudo x accipiatur infinita, exſurgit tempus t finitum; quum id autem non ſtatim appareat in praecedenti formula, ſic breviter oſtenditur: Quoniam ex hypotheſi $x = \infty$, & $c = \infty$, & praeterea $c = \surd(\varphi^2 + 2gr)$, erit etiam $\varphi = \infty$; hinc primus formulae terminus $\frac{1}{\varphi^2}\surd(r+x)(\varphi^2 x + rc^2)$ mutatur in $\frac{1}{\varphi^2}\surd\varphi^2 x^2 = \frac{x}{\varphi}$, terminus ſecundus $-\frac{rc}{\varphi^2}$ convertitur in $-\frac{r}{\varphi}$, numerus vero logarithmi termini tertii numeratorem habet $[\varphi\surd(r+x) - \surd(\varphi^2 x + rc^2)](\varphi + c) = \left(\varphi\surd(r+x) - \surd(\varphi^2 x + \varphi^2 r) - \frac{gr^2}{\surd(\varphi^2 x + rc^2)}\right) \times 2\varphi = -2gr^2\surd\frac{1}{x}$, habetque denominatorem $[\varphi\surd(r+x) + \surd(\varphi^2 x + rc^2)](\varphi - c) = 2\varphi\surd x(\varphi\surd r - \varphi\surd r - \frac{gr}{\varphi}) = -2gr\surd x$, adeoque numerus ipſe eſt $= -\frac{2gr^2\surd\frac{1}{x}}{-2gr\surd x} = \frac{r}{x}$. Igitur

$t = \frac{x}{\varphi} - \frac{r}{\varphi} + \frac{gr^2}{\varphi^3} \log. \frac{r}{x}$, ſeu ob valores infiniteſimos $\frac{r}{\varphi}$, & $\frac{gr^2}{\varphi^3} \log. \frac{r}{x}$, remanet $t = \frac{x}{\varphi}$, quantitati nimirum infinitae, ſiquidem x, & c ſeu φ valores habent infinitos ejuſdem ordinis.

15. Nunc operae pretium eſt ſelecto aliquo exemplo uſum formularum §. 10., & 11. illuſtrare. Petitur itaque tempus t, quo corpus ſurſum projectum velocitate $c = 2\sqrt{2gr}$, ſeu velocitate pedum paris. 69061, 378 ſingulis ſecundis ad ſexaginta ſemidiametrorum terreſtium altitudinem, ſeu ad Lunam uſque perveniret. Temporis expreſſio ſupra inventa dat $t = \frac{2gr^2 v}{h^2(v^2 - h^2)}$

$$- \frac{rc}{h^2} + \frac{gr^2}{h^3} \log. \frac{(v-h)(c+h)}{(v+h)(c-h)} =$$

$$\frac{1}{h^2} \sqrt{(r+x)(h^2 x + rc^2)} - \frac{rc}{h^2} + \frac{gr^2}{h^3} \times$$

$$\log. \frac{[h\sqrt{(r+x)} - \sqrt{(h^2 x + rc^2)}](h+c)}{[h\sqrt{(r+x)} + \sqrt{(h^2 x + rc^2)}](h-c)}.$$

Eſt jam

$g =$ 30, 2 ped.

$r =$ 19741200

$gr =$ 596184240

$c^2 = 8gr =$ 4769473920

$c = 2\sqrt{2gr} =$ 69061, 378

$h^2 = c^2 - 2gr =$ 3577105440

$x = 60r =$ 1184472000

$$v^2 = c^2 - \frac{2grx}{r+x} = 3596652464,\ 2623$$

$$v = \sqrt{\left(c^2 - \frac{2grx}{r+x}\right)} = 59972,\ 0974$$

$gr^2 v =$ 70583514247516857621I, 200

$\frac{1}{2} h^2 (v^2 - h^2) =$ 34960883412242658, 456

$$\frac{gr^2 v}{\frac{1}{2}h^2(v^2 - h^2)} = \frac{2gr^2 \cdot v}{h^2(v^2 - h^2)} = 20189,\ 28223$$

$$-\frac{rc}{h^2} = 381,\ 13343$$

$$\sqrt{(r+x)} = \quad 34701,\ 77517$$

$$\sqrt{(h^2 x + rc^2)} = \quad 2081138239,\ 8289$$

$$\sqrt{(r+x).(h^2 x + rc^2)} = \quad 72219191297746,\ 5584$$

$$\frac{1}{h^2}\sqrt{(r+x)(h^2 x + rc^2)} = \quad 20189,\ 28223$$



$$-\frac{rc}{h^2} = \quad -\ 381,\ 13343$$

$$h = \sqrt{(c^2 - 2gr)} = 59808,908$$

$$h^2 = 213942770167259,52$$

$$\frac{gr^2}{h^2} = \frac{11769392318688000}{213942770167259,52} = 55,01187$$

$$v - h = 163,1894$$

$$c + h = 128870,2860$$

$$(v - h)(c + h) = 21030264,65016840$$

$$v + h = 119781,0054$$

$$c - h = 9252,470$$

$$(v + h)(c - h) = 1108270159,03333800$$

$$\frac{(v - h)(c + h)}{(v + h)(c - h)} = 0,018975754$$

$$log.\ tab.\ \frac{(v - h)(c + h)}{(v + h)(c - h)} = -1,721801$$

$$log.hyp.\frac{(v-h)(c+h)}{(v+h)(c-h)} = -1,721801 \times 2,302585 = -3,964593$$

$$\frac{gr^2}{h^2}\, log.\frac{(v-h)(c+h)}{(v+h)(c-h)} = 55,01187 \times -3,964593 = -218,09967$$

$h\sqrt{(r+x)} =$ 2075475278, 5792

$\sqrt{(h^2x+rc^2)} =$ 2081138239, 8289

$h\sqrt{(r+x)} - \sqrt{(h^2x+rc^2)} = -$ 5662961, 2497

$h+c =$ 128870, 286

$[h\sqrt{(r+x)} - \sqrt{(h^2x+rc^2)}](h+c) =$

$= -$ 729787435855, 7564142

$h\sqrt{(r+x)} + \sqrt{(h^2x+rc^2)} =$ 4156613518, 4081

$h - c = -$ 9252, 470

$[h\sqrt{(r+x)} + \sqrt{(h^2x+rc^2)}](h-c) =$

$= -$ 38458941880665, 3930070

$$\frac{[h\sqrt{(r+x)} - \sqrt{(h^2x+rc^2)}](h+c)}{[h\sqrt{(r+x)} + \sqrt{(h^2x+rc^2)}](h-c)} = A$$

$$= \frac{729787435855,7564142}{38458941880665,3930070} = 0,018975754$$

$log.\ tab.\ A = -$ 1, 721801

$log.hyp.A = -1,721801 \times 2,320585 = -3,964593$

$\frac{gr^2}{h^3} log.A = 55,01187 \times -3,964593 = -218,09967$

Igitur oritur $t'' = 20189,28223 - 381,13343 - 218,09967 = 19590,0413'' = 5^h.\ 26'.\ 30''$ temporis ſiderei, ſeu $5^h.\ 24'.\ 52''$ temporis medii. Quare corpus celeritate pedum 69061, 378 pro unoquoque ſecundo ſurſum projectum, & a gravitate terreſtri decreſcente in ratione inverſa duplicata diſtantiarum jugiter retardatum horis 5, minutis 24, ſecundis 52 ad altitudinem ſexaginta ſemidiametrorum terreſtrium pertingit.

16. In hypotheſi gravitatis conſtantis, invenitur temporis expreſſio $t = \frac{c - v}{g} = \frac{c - \sqrt{(c^2 - 2gx)}}{g}$, quae factis ſubſtitutionibus exempli praecedentis apparet imaginaria: quod ſane indicat, corpus ea velocitate ſurſum emiſſum in hypotheſi gravitatis conſtantis non poſſe ad tantam altitudinem pervenire.



DISQUISITIO V.

De Sideribus intervallum inter datos duos Almicantarath interceptum velocissime trajicientibus, seu a data quamlibet altitudine ad aliam quamlibet datam tempore quamminimo pertingentibus.

1. IN Mathematicis inveſtigationibus aliquando, licet infrequenter experiuntur Geometrae, Syntheticam Methodum, hoc eſt Syntheſim Geometricam facilius ac tutius ad Problematis ſolutionem ducere quam ipſa Methodus Analytica, quae interdum per multiplices ſupputationum intricatiſſimarum moeandros id praeſtat, quod Syntheſis breviſſime & elegantiſſime abſolvit. At nihilominus nemo negaverit, directam regiamque viam, qua tendit Analyſis ad Verum ignotum aſſequendum, viae indirectae & obliquae, cui Syntheſis inhae-

ret, longe eſſe anteponendam. Inſignis certe Geometra Condorcetus *tom. III. Supplém. à l' Encyclopéd. Art. Méthode* non dubitat affirmare : *Les opérations de la Syntheſe ſont plus compliquées, ſa marche plus difficile à ſuivre, ſes réſultats moins généraux. Elle demanderoit pour bien des problêmes un travail impraticable : auſſi a-t-elle été abandonée de preſque tous le Géometres, & elle n'a plus pour elle que le nom de NEWTON, qui s'en ſervit, dit-on, pour cacher la route qu' il avoit ſuivie, & qui, sûr de l' admiration des grands Géometres, avoit la foibleſſe de vouloir encore étonner les eſprits médiocres. Mais je ne ſaurois être de cet avis, ſoit parce que cette petite charlatannerie me paroît trop indigne de ce grand-homme, ſoit parce qu' il* &c.

2. Hinc eſt, quod Geometrae recentiores magnam ſibi ſemper laudem pepererunt, ſi quandoque quaeſtionem aliquam Geometrica dumtaxat conſtructione ſeu Syntheſi definitam ad Analyſim feliciter revocarunt, ſeu ad generales formulas traduxerunt. Deſiderabat magnus EULERUS in immortali Opere *De Methodo inveniendi Lineas Curvas Maximi, Minimive proprietate gaudentes Cap. II.* §. 39. methodum aliquam a reſolutione geometrica & lineari omnino liberam, per quam analytica generalique expreſſione repraeſentari poſſet quantitas quaedam, quae per linearem dumtaxat con-

ſtructionem ab ipſo repraeſentata Opus illud eximium & cedro digniſſimum minus perfectum abſolutumque efficiebat. Cum ecce ſummus Geometra De la Grange invento *Variationum Calculo*, impletoque voto Euleriano (*a*) vix credibile dictu eſt, quantum inde nactus ſit celebritatis & gloriae.

3. Noviſſime eximius idem Geometra De la Grange *in Novis Actis Berolinenſis Academiae* pro anno 1773., editis anno 1775. jure meritoque ſibi plaudit, quod pervulgatum Problema De Sphaeroidum Ellipticarum Attractione, a ſagaciſſimo Maclaurino ingenioſiſſima Syntheſi jam pridem ſolutum, generali directaque methodo tractare, & ad analyſim transferre ſibi contigerit: hacque porro analytica ſolutione, tamquam argumento utitur adverſus Analyſis detractores, qui ut Syntheſim, quam unice excoluerunt, in Coelum ferant, Analyſim ſuggillant & carpunt. *Je me propoſe*, inquit, *dans ce Mémoire de faire voir que bien loin que le Problême dont il s'agit ſe refuſe à l'Analyſe, il peut être réſolu par ce moyen d'une manière, ſi non plus ſimple, du moins plus directe & plus générale*

(a) Vid. *Mélang. de Phil. & de Math. de la Soc. Roy. de Turin* tom. II. *Eſſai d'une Nouvelle Méthode pour déterminer les Maxima, & les Minima des Formules intégrales indéfinies, par Mr. De LA GRANGE.*

que par la voie de la Syntheſe ; ce qui ſervira à détruire un des principaux argumens que les détracteurs de l'Analyſe puiſſent apporter pour la rabaiſſer & pour prouver la ſupériorité de la méthode ſynthétique des Anciens.

4. Neque minorem conſequutus eſt laudem hic Cl. Geometra; quod in nobiliſſimo Problemate De Rotatione Corporis nulla vi acceleratrice ſollicitati quum binas ſolutiones Eulerianam, & Alembertianam minus directas ac generales obſervaſſet quippe quae notam aſſumunt poſitionem trium axium uniformis rotationis, methodum ipſe invenerit omnino directam, generalem, ac mere analyticam, per quam nulla aſſumpta Axium rotationis proprietate (haec enim ſolutionis eſſe debet conſectarium, non fundamentum) ad Problematis ſolutionem pervenitur (*b*). In ipſo autem Opuſculi

(*b*) Vid. *Nouvelle Solution du Problême du mouvement de rotation d'un corps de figure quelconque qui n'eſt animé par aucune force accélératrice*, in Actis Acad. Berol. tom. cit.

Recentiſſime EULERUS *in Novis Commentariis Acad. Petropolitanae* tom. XX. Diſſert. *De Nova Methodo Motum Corporum rigidorum determinandi*, idem Problema retractavit, & elegantiſſima analyſi directe ac generaliſſime reſolvit, demonſtrata prius inſigni ac pulcherrima corporum quorumque rigidorum proprietate, quod quomodocumque corpus rigidum ex ſtatu initiali in alium quemlibet ſtatum transferatur, in eo ſemper talis axis definiri poſſit, cujus directio in utroque ſtatu maneat invariata.

ingreſſu vere profitetur, *c'eſt toujours contribuer à l'avancement des Mathématiques que de montrer comment on peut reſoudre les mêmes queſtions & parvenir aux mêmes réſultats par des voies très différentes; les méthodes ſe prêtent par ce moyen un jour mutuel & en acquierent ſouvent un plus grand degré d'évidence & de généralité.*

5. Quum hujuſmodi conſiderationes ſe mihi identidem offerrent, animoque ſaepius obverſarentur, noviſſime factum eſt, ut tomum viceſimum nuper editum Novorum Academiae Peropolitanae Commentariorum evolvens in Aſtronomicam Diſſertationem inciderim illuſtris Euleri *De Trajectu citiſſimo Stellae per duos circulos Almicantarath datos pro qualibet elevatione Poli*; ubi Analyſta incomparabilis, repudiata praeter morem analyſi quaſi ad rem minus idonea, Problema aggreditur via ferme lineari ac ſynthetica, ſolutionemque nanciſcitur ſimplicem oppido & elegantem. Perculit me res nova & inaudita, magnum Eulerum in Problematis difficilis ſolutione Analyſi ſuae valedicere, ac Syntheſi inhaerere voluiſſe: ſed admirationem omnem abjeci cum in Euleriani Schediaſmatis concluſione haec mihi verba occurrerunt: *Hoc igitur modo ſine dubio ſolutionem ſimpliciſſimam problematis propoſiti ſumus nacti, quae, niſi hoc artificium fuiſſet adhibitum, in calculos vehementer prolixos praecipitaſſet.*

6. Hac ſummi Geometrae admonitione deterritus, & inextricabilem calculorum molem ab Eulero praenuntiatam reformidans vix audebam de directa & analytica tam pulchri Problematis ſolutione cogitare; ſed tanti Viri auctoritatem vicit cupiditas experiendi analyſis vires in Problemate, cujus analytica ſolutio ideo Eulero videri potuit immaniter operoſa & fere inacceſſa, quod per arduum & impeditum iter ad metam fortaſſe properaverit. At non una eſt ad veritatem via, neque ſemper contingit, ut via tutior ac brevior exercitatiſſimo cuique pateat. Quum itaque Eulerianum hoc Problema ad Analyſim revocare ſtatuiſſem, inſperanti mihi contigit, ut via ſatis expedita aequationem nanciſcerer biquadraticam, quam ſecus atque Eulerus pronunciavit tam evidenter aperteque reſolubilem reperi, ut nihil facilius, nihil ſimplicius expectari poſſe videretur. Ecce jam totius rei ſeriem & ordinem.

Fig. 4. 7. Circulus $NN'Q$ repraeſentat Horizontem loci; P Polum; Z Zenith; S & S' Sidus, paralelli arcum SS' inter duos datos Almicantarath interceptum citiſſime trajiciens. Ductis porro circulorum maximorum arcubus, quos ſchema exhibet, patet eſſe PQ altitudinem Poli ſeu loci latitudinem, ejuſque complementum PZ; SN altitudinem Sideris, & illius

complementum SZ; $S'N'$ altitudinem aliam Sideris, ejuſque complementum $S'Z$; PS, vel ſibi aequalem PS' eſſe complementum declinationis ejuſdem Sideris; denique ZPS angulum unum horarium, ZPS' alterum. Fiat modo.

Altitudo Aſtri	SN	. . .	$= a$
Altitudo alia	$S'N'$	. . .	$= a'$
Latitudo Terreſtris	PQ	. . .	$= l$
Aſtri declinatio		. . .	$= x$
Angulus Horarius	ZPS	. . .	$= h$
Angulus alter Horarius	ZPS'	. . .	$= h'$

8. Hoc facto manifeſtum eſt, fore

$$\sin.\ PZ \ldots\ldots = \cos.\ l$$
$$\sin.\ ZS \ldots\ldots = \cos.\ a$$
$$\sin.\ ZS' \ldots\ldots = \cos.\ a'$$
$$\sin.\ PS = \sin.\ PS' = \cos.\ x$$

rurſus

$$\cos.\ PZ \ldots\ldots = \sin.\ l$$
$$\cos.\ ZS \ldots\ldots = \sin.\ a$$
$$\cos.\ ZS' \ldots\ldots = \sin.\ a'$$
$$\cos.\ PS = \cos.\ PS' = \sin.\ x$$

Conſtat vero ex Trigonometria Sphaerica, in Triangulo ZPS hanc dari aequalitatem $\cos.\ ZPS$ $= \dfrac{\cos.\ ZS - \cos.\ PZ \cos.\ PS}{\sin.\ PZ \sin.\ PS}$, & in triangulo ZPS'

iſtam $cos.\ ZPS' = \frac{cos.ZS' - cos.PZ\,cos.PS'}{ſin.\ PZ\,ſin.\ PS'}$

9. Factis ſubſtitutionibus hae aequalitates nanciſcuntur hanc formam $cos.\ h = \frac{ſin.\ a - ſin.\ l\,ſin.\ x}{cos.l\,cos.x}$, $cos.h' = \frac{ſin.\ a' - ſin.\ l\,ſin.\ x}{cos.l.\ cos.\ x}$. Quoniam in hoc problemate quaeritur declinatio Sideris, quod breviſſimo tempore paralelli arcum SS' inter duos datos Almicantarath concluſum tranſcurrit, hujuſque temporis menſura eſt angulus $SPS' = ZPS' - ZPS = h' - h$; fiet idcirco $d.(h' - h) = dh' - dh \doteq 0$. Ut valores eliciam fluxionum dh' & dh, ſumo differentialia praedictarum aequalitatum, & invenio

$$-dh\ ſin.\ h = \frac{dx\,(ſin.\ a\ ſin.\ x - ſin.\ l)}{cos.\ l\,cos.^2\,x}, \&$$

$$-dh'\ ſin.\ h' = \frac{dx\,(ſin.\ a'\ ſin.\ x - ſin.\ l)}{cos.\ l\ cos.^2\ x};\ \text{proindeque}$$

$$-dh = \frac{dx\,(ſin.\ a\,ſin.\ x - ſin.\ l)}{ſin.\ h\ cos.\ l\,cos.^2\,x},$$

$$dh' = -\frac{dx\,(ſin.\ a'\,ſin.\ x - ſin.\ l)}{ſin.\ h'\ cos.\ l\,cos.^2\,x}.$$

10. Porro habetur $ſin.\, h = \frac{1}{cos.\, l\, cos.\, x} \times$

$$\sqrt{(2\, ſin. a\, ſin.\, l\, ſin.\, x - ſin.^2\, a - ſin.^2\, x + cos.^2\, l)}, \&$$

$$ſin.\, h' = \frac{1}{cos.\, l\, cos.\, x} \times$$

$$\sqrt{(2\, ſin.\, a'\, ſin.\, l\, ſin.\, x - ſin.^2\, a' - ſin.^2\, x + cos.^2\, l)}.$$

Igitur

$$dh' - dh = \frac{dx(ſin. a\, ſin. x - ſin.\, l)}{cos. x \sqrt{(2ſin. a\, ſin. l\, ſin. x - ſin^2 a - ſin.^2 x + cos.^2 l)}}$$

$$- \frac{dx(ſin.\, a'\, ſin.\, x - ſin.\, l)}{cos.\, x \sqrt{(2\, ſin.\, a'\, ſin.\, l\, ſin.\, x - ſin.^2\, a' - ſin.^2 x + cos.^2 l)}}$$

$= 0$. Quare $\frac{ſin.\, a\, ſin.\, x - ſin.\, l}{\sqrt{(2\, ſin. a\, ſin. l\, ſin. x - ſin.^2 a - ſin.^2 x + cos.^2 l)}}$

$$= \frac{ſin.\, a'\, ſin.\, x - ſin.\, l}{\sqrt{(2\, ſin.\, a'\, ſin. l\, ſin. x - ſin.^2\, a' - ſin.^2\, x + cos.^2\, l)}}.$$

11. Ab hac poſtrema aequatione elimino asymmetriam, & poſt debitas reductiones nanciſcor aequationem biquadraticam hanc

$$(\text{ſin.}^2 a - \text{ſin.}^2 \grave{a})\,\text{ſin.}^4 x + (2\,\text{ſin.}\,a\,\text{ſin.}^2 \grave{a}\,\text{ſin.}\,l$$
$$- 2\,\text{ſin.}\,\grave{a}\,\text{ſin.}^2 a\,\text{ſin.}\,l + 2\,\text{ſin.}\,\grave{a}\,\text{ſin.}\,l - 2\,\text{ſin.}\,a\,\text{ſin.}\,l)\,\text{ſin.}^3 x$$
$$- \cos.^2 l\,(\text{ſin.}^2 a - \text{ſin.}^2 \grave{a})\,\text{ſin.}^2 x - (2\,\text{ſin.}\,a\,\text{ſin.}^2 \grave{a}\,\text{ſin.}\,l$$
$$- 2\,\text{ſin.}\,\grave{a}\,\text{ſin.}^2 a\,\text{ſin.}\,l + 2\,\text{ſin.}\,\grave{a}\,\text{ſin.}\,l - 2\,\text{ſin.}\,a\,\text{ſin.}\,l)\,\text{ſin.}\,x$$
$$+ \text{ſin.}^2 \grave{a}\,\text{ſin.}^2 l - \text{ſin.}^2 a\,\text{ſin.}^2 l = 0.$$

In hac autem aequatione ſtatim apparet, coefficientem ſecundi termini quadrinomium contrahi poſſe in binomium $-(2\,\text{ſin.}\,a\cos.^2 \grave{a} - 2\,\text{ſin.}\,\grave{a}\cos.^2 a)\,\text{ſin.}\,l$, & coefficientem quarti in binomium oppoſitum $(2\,\text{ſin.}\,a \times \cos.^2 \grave{a} - 2\,\text{ſin.}\,\grave{a}\cos.^2 a)\,\text{ſin.}\,l$. Quapropter diviſa aequatione per $\text{ſin.}^2 a - \text{ſin.}^2 \grave{a}$, invenitur

$$\text{ſin.}^4 x - \frac{(2\,\text{ſin.}\,a\cos.^2 \grave{a} - 2\,\text{ſin.}\,\grave{a}\cos.^2 a)\,\text{ſin.}\,l}{\text{ſin.}^2 a - \text{ſin.}^2 \grave{a}}\,\text{ſin.}^3 x$$
$$- \cos.^2 l\,\text{ſin.}^2 x + \frac{(2\,\text{ſin.}\,a\cos.^2 \grave{a} - 2\,\text{ſin.}\,\grave{a}\cos.^2 a)\,\text{ſin.}\,l}{\text{ſin.}^2 a - \text{ſin.}^2 \grave{a}}\,\text{ſin.}\,x$$
$$- \text{ſin.}^2 l = 0.$$

12. Jam vero aequationem hanc vel oſcitanter conſideranti ſtatim occurrit, utramque hypotheſim $\text{ſin.}\,x = 1$, & $\text{ſin.}\,x = -1$ ipſi aequationi ſatisfacere, ac proinde $\text{ſin.}^2 x - 1$ eſſe illius diviſorem. Quamobrem diviſione abſoluta oritur quadratica aequatio

$$\text{ſin.}^2 x - \frac{(2\,\text{ſin.}\,a\cos.^2\grave{a} - 2\,\text{ſin.}\,\grave{a}\cos.^2 a)\,\text{ſin.}\,l}{\text{ſin.}^2 a - \text{ſin.}^2 \grave{a}}\,\text{ſin.}\,x$$

$+\,\text{ſin.}^2 l = 0$, cujus radix eſt $\text{ſin.}\,x = \dfrac{\text{ſin.}\,a\cos.^2\grave{a} - \text{ſin.}\,\grave{a}\cos.^2 a}{\text{ſin.}^2 a - \text{ſin.}^2\grave{a}}\,\text{ſin.}\,l$

$$\pm\,\text{ſin.}\,l\,\sqrt{\left(\left(\frac{\text{ſin.}\,a\cos.^2\grave{a} - \text{ſin.}\,\grave{a}\cos.^2 a}{\text{ſin.}^2 a - \text{ſin.}^2\grave{a}}\right)^2 - 1\right)}.$$

13. Itaque radices omnes aequationis ſuperioris habemus in poteſtate, nempe

1. $\text{ſin.}\,x = 1$

2. $\text{ſin.}\,x = -1$

3. $$\text{ſin.}\,x = \frac{\text{ſin.}\,a\cos.^2\grave{a} - \text{ſin.}\,\grave{a}\cos.^2 a}{\text{ſin.}^2 a - \text{ſin.}^2\grave{a}}\,\text{ſin.}\,l$$

$$+\,\text{ſin.}\,l\,\sqrt{\left(\left(\frac{\text{ſin.}\,a\cos.^2\grave{a} - \text{ſin.}\,\grave{a}\cos.^2 a}{\text{ſin.}^2 a - \text{ſin.}^2\grave{a}}\right)^2 - 1\right)}$$

4. $$\text{ſin.}\,x = \frac{\text{ſin.}\,a\cos.^2\grave{a} - \text{ſin.}\,\grave{a}\cos.^2 a}{\text{ſin.}^2 a - \text{ſin.}^2\grave{a}}\,\text{ſin.}\,l$$

$$-\,\text{ſin.}\,l\,\sqrt{\left(\left(\frac{\text{ſin.}\,a\cos.^2\grave{a} - \text{ſin.}\,\grave{a}\cos.^2 a}{\text{ſin.}^2 a - \text{ſin.}^2\grave{a}}\right)^2 - 1\right)}$$

14. Expendamus jam paullo diligentius harum radicum indolem, & quae Problemati ſolvendo pares

ſint, quaeve haudquaquam idoneae indigitemus. Ac primo liquet, radices priores duas $ſin. x = 1$, $ſin. x = -1$ ad rem noſtram non eſſe; nam Sidera integro quadrante Boream, & Auſtrum verſus ab Æquatore remota, atque adeo Polum utrumque occupantia nec detecta unquam ſunt, nec diurna, ſi exiſterent, raperentur vertigine, quae tamen Problematis conſtituit hypotheſim.

15. Excutienda nunc reſtat tertia radix

$$ſin.\ x = \frac{ſin.\ a\ cos.^2\ \grave{a} - ſin.\ \grave{a}\ cos.^2\ a}{ſin.^2\ a - ſin.^2\ \grave{a}}\ ſin.\ l$$

$$+ ſin.\ l \sqrt{\left(\left(\frac{ſin.\ a\ cos.^2\ \grave{a} - ſin.\ \grave{a}\ cos.^2\ a}{ſin.^2\ a - ſin.^2\ \grave{a}}\right)^2 - 1\right)}.$$

Adverto itaque ſtatim, aequationem ſuperiorem biquadraticam produci tum ex poſitione $dh' - dh = 0$, tum ex poſitione $dh' + dh = 0$; nam in aequalitatibus $dh' = dh$, & $dh' = -dh$ quadratis utrinque membris ſignorum oppoſitio evaneſcit, habeturque $dh'^2 = dh^2$, atque inde biquadratica aequatio §. 11. Porro tertia haec radix $ſin.\ x$ ſpectat ad poſitionem $dh' + dh = 0$, minime vero ad poſitionem noſtram $dh' - dh = 0$, adeoque in hoc problemate, quod nititur hypotheſi $dh' - dh = 0$, eſt omnino rejicienda. Id demonſtrabitur, ſi in

expreſſione $dh + dh'$. ſubſtituto valore hujus radicis quantitas producatur evidenter nihilo aequalis; quod ſane contingere, ita oſtendo: Ponamus demonſtrationis contrahendae gratia, Almicantarath alterutrum cum Horizonte congruere, ita ut ipſius altitudo a ſit $= 0$; oriturque in hac hypotheſi tertia haec radix $\text{ſin.}\, x$

$$= \frac{\text{ſin.}\, l}{\text{ſin.}\, a} + \text{ſin.}\, l \sqrt{\left(\frac{1}{\text{ſin.}^2 a} - 1\right)} = \frac{\text{ſin.}\, l + \text{ſin.}\, l \cos.\, a}{\text{ſin.}\, a}.$$

Loco $\text{ſin.}\, x$ ſubſtituo hunc valorem in expreſſione $dh + dh'$ §. 10., & conſequor $dh + dh'$

$$= -dx\, \text{ſin.}\, l \cos.\, a : \left(\cos.\, x \sqrt{\left(2\, \text{ſin.}^2 l + 2\, \text{ſin.}^2 l \cos.\, a \right.}\right.$$

$$\left.\left. - \text{ſin.}^2 a + \cos.^2 l - \frac{\text{ſin.}^2 l}{\text{ſin.}^2 a} - \frac{2\, \text{ſin.}^2 l \cos.\, a}{\text{ſin.}^2 a} - \frac{\text{ſin.}^2 l \cos.^2 a}{\text{ſin.}^2 a}\right)\right)$$

$$+ dx\, \text{ſin.}\, l : \left(\cos.\, x \sqrt{\left(\cos.^2 l - \frac{\text{ſin.}^2 l}{\text{ſin.}^2 a}\right.}\right.$$

$$\left.\left. - \frac{2\, \text{ſin.}^2 l \cos.\, a}{\text{ſin.}^2 a} - \frac{\text{ſin.}^2 l \cos.^2 a}{\text{ſin.}^2 a}\right)\right);$$

& ſi primum radicale dicatur $\sqrt{A}$, alterum $\sqrt{B}$ fiet $dh + dh' = -\dfrac{dx\, \text{ſin.}\, l \cos.\, a \sqrt{B} + dx\, \text{ſin.}\, l \sqrt{A}}{\cos.^2 x \sqrt{AB}}$.

Eſt autem $\cos.\, a \sqrt{B} = \sqrt{(B - B\, \text{ſin.}^2 a)} = \sqrt{A}$. Igitur $dh + dh' = 0$. Quapropter tertia haec radix

aliam reſpicit hypotheſim a noſtra diverſam, nec eſt proinde attendenda.

16. Sola ergo ſupereſt quarta radix

$$ſin.\ x = \frac{ſin.\ a\ cos.^2\ \dot{a} - ſin.\ \dot{a}\ cos.^2\ a}{ſin.^2\ a - ſin.^2\ \dot{a}}\ ſin.\ l$$

$$- ſin.\ l \sqrt{\left(\left(\frac{ſin.\ a\ cos.^2\ \dot{a} - ſin.\ \dot{a}\ cos.^2\ a}{ſin.^2 a - ſin.^2 \dot{a}}\right)^2 - 1\right)},$$

quae Problematis ſolutionem praebet. Hic autem primo occurrit, in regionibus ſub aequatore ſitis, ubi ſcilicet $l = 0$, ſidera in aequatore ipſo conſtituta iter illud citiſſimum conficere; in hac quippe hypotheſi invenitur $ſin.\ x = 0$: in regionibus vero reliquis citra vel ultra aequatorem ſidera citiſſimi trajectus declinationem ſemper aliquam habent, quandoquidem ſi nullam haberent, hoc eſt ſi $ſin.x$, ſeu $\frac{ſin.a\ cos.^2\ \dot{a} - ſin.\dot{a}\ cos.^2 a}{ſin.^2\ a - ſin.^2\ \dot{a}}\ ſin.l$

$$- ſin.\ l \sqrt{\left(\left(\frac{ſin.a\ cos.^2\ \dot{a} - ſin.\ \dot{a}\ cos.^2\ a}{ſin.^2\ a - ſin.^2\ \dot{a}}\right)^2 - 1\right)}$$

foret nihilo aequalis, prodiret $- ſin.^2 l = 0$, quod eſt abſurdum.

17 Si altitudo a accipiatur $= 90°$, & ſidera quaerantur, quorum aſcenſus ab altitudine altera $\dot{a}$ ad Zenith ſit omnium celerrimus, invenitur $ſin.x = ſin.l$;

nam, quum in hac hypotheſi ſit $ſin.\ a = 1$, $cos.\ a = 0$, evaneſcit quantitas radicalis, ac remanet $ſin.\ x = ſin.\ l$, adeoque $x = l$. Quocirca ſidera, quorum declinatio eſt Poli elevationi aequalis, citiſſime ab altitudine qualibet data ad Zenith progrediuntur. Sed unicum eſt in hac hypotheſi *minimum tempus*, nec proinde cum aliis conferendum, ſiquidem una tantum eſt declinatio Aſtrorum per Zenith tranſeuntium. Idem reperitur, ſi facto ut prius $a = 90°$, ſumatur $\grave{a} = 0$, ſeu quaeratur declinatio ſiderum ab horizonte ad zenith celerrime aſcendentium.

18. Congruente alterutro Almicantarath cum Horizonte, ſeu facto $\grave{a} = 0$, atque hinc $ſin.\ \grave{a} = 0$, $cos.\ \grave{a} = 1$, oritur $ſin.\ x = \frac{ſin.\ l - ſin.\ l\ cos.\ a}{ſin.\ a}$

$= ſin.\ l\ \frac{1 - cos.\ a}{ſin.\ a}$. Liquet autem, eſſe $tang.\ \frac{1}{2}\ a$

$= \frac{1 - cos.\ a}{ſin.\ a}$; propterea fit $ſin.\ x = ſin.\ l.\ tang.\ \frac{1}{2}\ a$. Eſt porro $tang.\ \frac{1}{2} a$ ſemper unitate minor praeter unicum caſum, quo eſt $a = 90°$, & ideo $tang.\ \frac{1}{2} a = 1$! igitur ſidera, quae ab Horizonte ad datam altitudinem breviſſimo tempore attolluntur, minus ab aequatore diſtant quam locus terreſtris, eaque tantum, quae Zenith diurna vertigine attingunt, eandem habent ab

aequatore diſtantiam ac regio terreſtris.

19. Minimum autem tempus, quo ſidera ab Horizonte ad datam altitudinem perveniunt, invenitur in hunc modum: Ex §. 9. eſt $\mathit{ſin.}\, h' (\mathit{ſin.} a\, \mathit{ſin.} x - \mathit{ſin.}\, l) + \mathit{ſin.}\, h\, \mathit{ſin.}\, l = 0$; adeoque ſubſtituto valore $\mathit{ſin.}\, x$, fit $\mathit{ſin}\, h = \mathit{ſin}\, h' \cos. a$. Rurſus $\cos. h' = -\frac{\mathit{ſin.}\, l\, \mathit{ſin.}\, x}{\cos. l \cos. x}$, & $\cos. h = \frac{\mathit{ſin.}\, a}{\cos. l. \cos. x} + \cos. h'$. Igitur $\cos. (h' - h)$

$$= \cos. h' \cos. h + \mathit{ſin.}\, h'\, \mathit{ſin}\, h = \frac{\mathit{ſin.}\, a \cos. h'}{\cos. l. \cos. x}$$

$$+ \cos.^2 h' + \cos. a\, \mathit{ſin.}^2 h' = \frac{\mathit{ſin.}\, a \cos. h'}{\cos. l \cos. x} + \cos.^2 h'$$

$-\cos. a \cos.^2 h' + \cos. a$, ſeu ſubrogato valore $\cos. h'$

$= -\frac{\mathit{ſin.}\, l. \mathit{ſin.}\, x}{\cos. l. \cos. x}$, & $\cos.^2 h' = \frac{\mathit{ſin.}^2 l\, \mathit{ſin.}^2 x}{\cos.^2 l \cos.^2 x}$, prodit

$$\cos.(h' - h) = \cos. a + \frac{[(1 - \cos. a)\mathit{ſin.} l. \mathit{ſin.} x - \mathit{ſin.} a]\, \mathit{ſin.} l\, \mathit{ſin.} x.}{\cos.^2 l \cos.^2 x}$$

Eſt porro §. 18. $(1 - \cos. a)\, \mathit{ſin.}\, l = \mathit{ſin.}\, a\, \mathit{ſin.}\, x$;

propterea fiet $\cos.(h' - h) = \cos. a + \frac{\mathit{ſin.} a\, \mathit{ſin.} l\, \mathit{ſin.} x (\mathit{ſin.}^2 x - 1)}{\cos.^2 l \cos.^2 x}$

$$= \cos. a - \frac{\mathit{ſin.} a\, \mathit{ſin.} l\, \mathit{ſin.} x}{\cos.^2 l} = \cos. a - \mathit{ſin.}\, a\, \mathit{tang.}^2 l\, \mathit{tang.} \tfrac{1}{2} a.$$

Quum autem conſtet, eſſe $tang. \frac{1}{2} a$

$$= \sqrt{\frac{1 - cos. a}{1 + cos. a}} = \frac{\sqrt{(1 - cos.^2 a)}}{1 + cos. a} = \frac{ſin. a}{1 + cos. a},$$

atque inde $ſin. a \; tang. \frac{1}{2} a = \frac{ſin.^2 a}{1 + cos. a} = \frac{1 - cos.^2 a}{1 + cos. a}$

$= 1 - cos. a = 2 ſin.^2 \frac{1}{2} a$; oritur iccirco $cos. (h' - h)$
$= cos. a - 2 \; tang.^2 l \; ſin.^2 \frac{1}{2} a = 1 - 2 ſin.^2 \frac{1}{2} a$
$- 2 \; tang.^2 l \; ſin.^2 \frac{1}{2} a = 1 - 2 ſin.^2 \frac{1}{2} a \, (1 + tang.^2 l)$

$= 1 - 2 ſin.^2 \frac{1}{2} a \; ſec.^2 l = 1 - \frac{2 ſin.^2 \frac{1}{2} a}{cos.^2 l}$. Inventa

eſt igitur expreſſio ſimplex & concinna $1 - \frac{2 ſin.^2 \frac{1}{2} a}{cos.^2 l}$

pro coſinu illius arcus $h' - h$, qui tempus quaeſitum repraeſentat. Sed ſimplicior adhuc reddi poteſt hujuſmodi formula; haec enim fit $\frac{2 ſin.^2 \frac{1}{2} a}{cos.^2 l} =$

$1 - cos. (h' - h) = 2 ſin.^2 \frac{1}{2} (h' - h)$. Quocirca $ſin. \frac{1}{2} (h' - h)$

$= \frac{ſin. \frac{1}{2} a}{cos. l}$, expreſſio ſimpliciſſima ſemiarcus tempus

dimidium propoſitum metientis.

20. Si dimidium datae altitudinis aequetur complemento elevationis Poli, hoc eſt $\frac{1}{2} a = 90° - l$,

invenitur declinatio Aſtri citiſſime aſcendentis aequalis ejuſdem altitudinis dimidio: nam $\mathit{ſin.}\,x = \mathit{ſin.}\,l\,\mathit{tang.}\,\tfrac{1}{2}\,a = \mathit{ſin.}\,\tfrac{1}{2}\,a$. Tempus vero aſcenſus Aſtri ab horizonte ad altitudinem a elicitur ex formula $\mathit{ſin.}\,\tfrac{1}{2}\,(h' - h)$

$$= \frac{\mathit{ſin.}\,\tfrac{1}{2}\,a}{\mathit{cos.}\,l} = \frac{\mathit{ſin.}\,\tfrac{1}{2}\,a}{\mathit{ſin.}\,\tfrac{1}{2}\,a} = 1;$$ unde oritur $\tfrac{1}{2}\,h' - \tfrac{1}{2}\,h = 90°$, & $h' - h = 180°$, ex quo habentur $12.^{h.}$ temporis ſiderei, vel $11.^{h.}\,58'.\,2''$ temporis medii. In hac enim hypotheſi parallelus ab Aſtro deſcriptus tangitur ab horizonte & almicantarath in binis oppoſitis punctis, ſeu ſemiperipheria diſtantibus. Si autem dimidium altitudinis datae excedat complementum elevationis Poli, nanciſcimur temporis expreſſionem omnino repugnantem, ſeu $\mathit{ſin.}\,\tfrac{1}{2}\,(h' - h) = \frac{\mathit{ſin.}\,\tfrac{1}{2}\,a}{\mathit{cos.}\,l} > 1$, quod eſt abſurdum. Sane in hac hypotheſi nullum eſt Aſtrum, cujus parallelus horizonti ſimul & dato almicantarath occurrat; paralleli enim omnes horizontem ſecantes vel tangentes ad almicantarath non protenduntur, & qui almicantarath ſecant vel tangunt ad horizontem non pertingunt.

21. In regionibus aequatoriis ſidera, quae omnium velociſſime tranſeunt ab uno ad alterum almicantarath, ſunt ea, quae aequatorem ipſum deſcribunt ſeu quo-

rum nulla eſt declinatio; nam in formula §. 16. ſumpto $l = 0$, oritur $ſin.\ x = 0$. Tempus autem hujuſce tranſitus eſt $= a - \grave{a}$; quum in hac hypotheſi ſint a, & $\grave{a}$ arcus aequatoris per zenith tranſeuntis, ſimulque complementa angulorum, vel arcuum h', & h; ac proinde $h' - h = a - \grave{a}$.

22. Ad rei totius illuſtrationem operae pretium eſt exemplum afferre, & formulae §. 16. evolutionem ob oculos ponere. Quaeritur itaque in hac Urbe Ticini ejus Sideris declinatio quod breviſſimo tempore ab altitudine $\grave{a}$ 20° ad altitudinem a 50° pertingit. En calculi typum.

$\grave{a}$	$= 20°$
a	$= 50°$
l	$= 45°.\ 11'$
$log.\ cos.\ \grave{a}$	$= 9,\ 9729858$
$log.\ cos.^2\ \grave{a}$	$= 9,\ 9459716$
$log.\ ſin.\ a$	$= 9,\ 8842540$
$log.\ ſin.\ a\ cos.^2\ \grave{a}$	$= 9,\ 8302256$
$ſin.\ a\ cos.^2\ \grave{a}$	$= 0,\ 676434$
$log.\ cos.\ a$	$= 9,\ 8080675$
$log.\ cos.^2\ a$	$= 9,\ 6161350$

$\log.\ \text{ſin}.\ \grave{a}$ . . . . $= 9,\ 5340517$

$\log.\ \text{ſin}.\ \grave{a}\ \cos.^2\ a$ . . . . . $= 9,\ 1501867$

$\text{ſin}.\ \grave{a}\ \cos.^2\ a$ . . . . $= 0,\ 1413145$

$\text{ſin}.\ a\ \cos.^2\ \grave{a} - \text{ſin}.\ \grave{a}\ \cos.^2\ a$ . . $= 0,\ 5351195$

$\log.\ \text{ſin}.^2\ a$ . . . . $= 9,\ 7685080$

$\text{ſin}.^2\ a$ . . . . $= 0,\ 586824$

$\log.\ \text{ſin}.^2\ \grave{a}$ . . . . $= 9,\ 0681034$

$\text{ſin}.^2\ \grave{a}$ . . . . $= 0,\ 116978$

$\text{ſin}.^2\ a - \text{ſin}.^2\ \grave{a}$ . . . . . $= 0,\ 469846$

$\log.\ (\text{ſin}.\ a \cos.^2\ \grave{a} - \text{ſin}.\ \grave{a} \cos.^2\ a)$. $= 9,\ 7284508$

$\log.\ (\text{ſin}.^2\ a - \text{ſin}.^2\ \grave{a})$ . . . $= 9,\ 6719555$

$\log. \dfrac{\text{ſin}.\ a \cos.^2\ \grave{a} - \text{ſin}.\ \grave{a} \cos.^2\ a}{\text{ſin}.^2\ a - \text{ſin}.^2\ \grave{a}}$. $= 0,\ 0564953$

$\log.\ \text{ſin}.\ l$ . . . . $= 9,\ 8508702$

$\log. \dfrac{\text{ſin}.\ a \cos.^2\ \grave{a} - \text{ſin}.\ \grave{a} \cos.^2\ a}{\text{ſin}.^2\ a - \text{ſin}.^2\ \grave{a}}\ \text{ſin}.\ l = 9,\ 9073655$

$\dfrac{\text{ſin}.\ a \cos.^2\ \grave{a} - \text{ſin}.\ \grave{a} \cos.^2\ a}{\text{ſin}.^2\ a - \text{ſin}.^2\ \grave{a}}\ \text{ſin}.\ l$. $= 0,\ 807915$

$\log. \left(\dfrac{\text{ſin}.\ a \cos.^2\ \grave{a} - \text{ſin}.\ \grave{a} \cos.^2\ a}{\text{ſin}.^2\ a - \text{ſin}.^2\ \grave{a}}\right)^2 = 0,\ 1129906$

$$\left(\frac{ſin.\,a\cos.^2 \grave{a} - ſin.\,\grave{a}\cos.^2 a}{ſin.^2 a - ſin.^2 \grave{a}}\right)^2 \ldots\ldots 1,29715$$

$$\left(\frac{ſin.\,a\cos.^2 \grave{a} - ſin.\,\grave{a}\cos.^2 a}{ſin.^2 a - ſin.^2 \grave{a}}\right)^2 - 1 \ldots\ldots 0,29715$$

$$log.\left(\left(\frac{ſin.\,a\cos.^2 \grave{a} - ſin.\,\grave{a}\cos.^2 a}{ſin.^2 a - ſin.^2 \grave{a}}\right)^2 - 1\right) \ldots 9,4729757$$

$$log.\sqrt{\left(\left(\frac{ſin.\,a\cos.^2 \grave{a} - ſin.\,\grave{a}\cos.^2 a}{ſin.^2 a - ſin.^2 \grave{a}}\right)^2 - 1\right)} \ldots 9,7364878$$

$$log.ſin.l\sqrt{\left(\left(\frac{ſin.\,a\cos.^2 \grave{a} - ſin.\,\grave{a}\cos.^2 a}{ſin.^2 a - ſin.^2 \grave{a}}\right)^2 - 1\right)} \ldots 9,5873580$$

$$ſin.l\sqrt{\left(\left(\frac{ſin.\,a\cos.^2 \grave{a} - ſin.\,\grave{a}\cos.^2 a}{ſin.^2 a - ſin.^2 \grave{a}}\right)^2 - 1\right)} \ldots 0,386686$$

$$ſin.\,x \ldots\ldots\ldots\ldots 0,421229$$

Igitur ſideris declinatio quaeſita $x = 24°.\ 54'.\ 44''$

23. Res nunc poſtulat, ut formula §. 16. valorem *ſin.* x repraeſentans brevior, ſimplicior, & ad uſum magis idonea ac parata reddatur. Hoc autem ita conficio: Notum eſt, dato quovis angulo φ eſſe $ſin.\tfrac{1}{2}\varphi$

$$= \sqrt{\frac{1 - \cos.\varphi}{2}},\ \&\ \cos.\tfrac{1}{2}\varphi = \sqrt{\frac{1 + \cos.\varphi}{2}}.$$ Igi-

tur $\text{ſin.}\,\frac{1}{2}(a+\grave{a}) = \sqrt{\frac{1-\text{cos.}(a+\grave{a})}{2}}$, &

$\text{cos.}\,\frac{1}{2}(a-\grave{a}) = \sqrt{\frac{1+\text{cos.}(a-\grave{a})}{2}}$; atque inde

$$\frac{\text{ſin.}\,\frac{1}{2}(a+\grave{a})}{\text{cos.}\,\frac{1}{2}(a-\grave{a})} = \sqrt{\frac{1-\text{cos.}(a+\grave{a})}{1+\text{cos.}(a-\grave{a})}}$$

$$= \sqrt{\frac{1+\text{ſin.}\,a\,\text{ſin.}\,\grave{a}-\text{cos.}\,a\,\text{cos.}\,\grave{a}}{1+\text{ſin.}\,\grave{a}\,\text{ſin.}\,a+\text{cos.}\,a\,\text{cos.}\,\grave{a}}}$$

$$= \frac{1+\text{ſin.}\,a\,\text{ſin.}\,\grave{a}-\text{cos.}\,a\,\text{cos.}\,\grave{a}}{\sqrt{[(1+\text{ſin.}\,a\,\text{ſin.}\,\grave{a})^2-\text{cos.}^2\,a\,\text{cos.}^2\,\grave{a}]}}$$

$$= \frac{1+\text{ſin.}\,a\,\text{ſin.}\,\grave{a}-\text{cos.}\,a\,\text{cos.}\,\grave{a}}{\sqrt{(1+2\,\text{ſin.}\,a\,\text{ſin.}\,\grave{a}+\text{ſin.}^2\,a\,\text{ſin.}^2\,\grave{a}-\text{cos.}^2\,a\,\text{cos.}^2\,\grave{a})}}$$

$$= \frac{1+\text{ſin.}\,a\,\text{ſin.}\,\grave{a}-\text{cos.}\,a\,\text{cos.}\,\grave{a}}{\text{ſin.}\,a+\text{ſin.}\,\grave{a}}$$

$$= \frac{(\text{ſin.}\,a-\text{ſin.}\,\grave{a})(1+\text{ſin.}\,a\,\text{ſin.}\,\grave{a}-\text{cos.}\,a\,\text{cos.}\,\grave{a})}{\text{ſin.}^2\,a-\text{ſin.}^2\,\grave{a}}$$

$$= \frac{\text{ſin.}\,a\,\text{cos.}^2\,\grave{a}-\text{ſin.}\,\grave{a}\,\text{cos.}^2\,a}{\text{ſin.}^2\,a-\text{ſin.}^2\,\grave{a}} - \frac{\text{cos.}\,a\,\text{cos.}\,\grave{a}\,(\text{ſin.}\,a-\text{ſin.}\,\grave{a})}{\text{ſin.}^2\,a-\text{ſin.}^2\,\grave{a}}.$$

Eſt porro quantitas $\frac{\text{cos.}\,a\,\text{cos.}\,\grave{a}\,(\text{ſin.}\,a-\text{ſin.}\,\grave{a})}{\text{ſin.}^2\,a-\text{ſin.}^2\,\grave{a}}$

$$= \frac{\cos. \grave{a} \; (\text{ſin}. \; a - \text{ſin}. \; \grave{a})}{\text{ſin}.^2 \; a - \text{ſin}.^2 \; \grave{a}} \sqrt{(1 - \text{ſin}.^2 \; a)}$$

$$= \sqrt{\frac{\cos.^2 \grave{a} \, (\text{ſin}.^2 a - 2\,\text{ſin}.a\,\text{ſin}.\grave{a} + \text{ſin}.^2 \grave{a})(1 - \text{ſin}^2 a)}{(\text{ſin}.^2 \; a - \text{ſin}.^2 \; \grave{a})^2}},$$

quae poſtrema expreſſio per idoneas reductiones convertitur in $\sqrt{\left(\left(\frac{\text{ſin}.\,a\cos.^2\grave{a} - \text{ſin}.\,\grave{a}\cos.^2 a}{\text{ſin}.^2\, a - \text{ſin}.^2\, \grave{a}}\right)^2 - 1\right)}$.

Igitur poſtremo $\text{ſin}. \; x = \frac{\text{ſin}.a\cos.^2\grave{a} - \text{ſin}.\grave{a}\cos.^2 a}{\text{ſin}.^2\, a - \text{ſin}.^2\, \grave{a}}\,\text{ſin}.l$

$$- \text{ſin}. \; l \sqrt{\left(\left(\frac{\text{ſin}.a\cos.^2\grave{a} - \text{ſin}.\grave{a}\cos.^2 a}{\text{ſin}.^2\, a - \text{ſin}.^2\, \grave{a}}\right)^2 - 1\right)}$$

$= \frac{\text{ſin}.l\,\text{ſin}.\frac{1}{2}(a + \grave{a})}{\cos.\frac{1}{2}(a - \grave{a})}$; quae ſane expreſſio ſimplicitate & elegantia cum ſimpliciſſimis & elegantiſſimis certat.

24. Eapropter *ſinus declinationis quaeſitae eſt quartus proportionalis ad coſinum ſemidifferentiae datarum altitudinum, vel ſemidiſtantiae binorum Almicantarath, ad ſinum ſemiſummae earundem altitudinum, & ad ſinum latitudinis terreſtris.* Atque ideo in ſuperiori exemplo §. 22. habebitur

log. ſin. l	9, 8508702
log. ſin. $\frac{1}{2}(a + \grave{a})$	9, 7585913
log. ſin. l ſin. $\frac{1}{2}(a + \grave{a})$	9, 6094615
log. cos. $\frac{1}{2}(a - \grave{a})$	9, 9849438
log. ſin. x	9, 6245177

Igitur $x = 24.° 54'. 44''$, prorſus idem ac ſupra §. 22.

25. Inventa declinatione Sideris breviſſimo tempore per binos datos Almicantarath tranſeuntis, invenitur etiam tempus ipſius tranſitus, hoc eſt arcus $h' - h$ in hunc modum: Per §. 9. eſt $\cos. h = \frac{\text{ſin.}\, a}{\cos. l \cos. x} - \text{tang.}\, l\, \text{tang.}\, x$, & $\cos. h' = \frac{\text{ſin.}\, \grave{a}}{\cos. l \cos. x} - \text{tang.}\, l\, \text{tang.}\, x$. Ergo dabitur tum $\cos. h$, tum $\cos. h'$; proindeque etiam h, & h'. Quocirca angulus $h' - h$ datus pariter erit, qui ſtatim in tempus convertitur. Sic in exemplo praecedenti

log. ſin. a	9, 8842540
log. cos. l	9, 8480909
log. cos. x	9, 9575854
log. cos. l cos. x	9, 8056763

$\log.\ \frac{\text{ſin}.\,a}{\cos.\,l\ \cos.\,x}$	0, 0785777
$\frac{\text{ſin}.\,a}{\cos.\,l\ \cos.\,x}$	1, 1983336
$\log.\ \text{tang}.\ l$	0, 0027793
$\log.\ \text{tang}.\ x$	9, 6669332
$\log.\ \text{tang}.\,l\ \text{tang}.\,x$	9, 6697125
$-\ \text{tang}.\ l\ \text{tang}\ x$	$-$0, 4674256
$\frac{\text{ſin}.\,a}{\cos.\,l\ \cos.\,x} - \text{tang}.\,l\,\text{tang}.\,x = \cos.\,h$.	0, 7309080
h	43° 2'. 15"
$\log.\ \text{ſin}.\ \grave{a}$	9, 5340517
$\log.\ \cos.\,l\ \cos.\,x$	9, 8056763
$\log.\ \frac{\text{ſin}.\,\grave{a}}{\cos.\,l.\ \cos.\,x}$	9, 7283754
$\frac{\text{ſin}.\,\grave{a}}{\cos.\,l\ \cos.\,x}$	0, 5350266
$-\ \text{tang}.l\ \text{tang}.x$	$-$0, 4674256
$\frac{\text{ſin}.\,\grave{a}}{\cos.\,l\ \cos.\,x} - \text{tang}.l\,\text{tang}.x = \cos.\,h'$.	0, 0676010
h'	86°. 7'. 26"

$h' - h$ . . . . 43.° 5.' 11."

Facta hujuſce anguli $h' - h$ converſione in tempus, prodeunt 2.$^{hor.}$ 52.$^{min.}$ 21.$^{sec.}$ temporis ſiderei, ſeu 2.$^{h.}$ 51.$^{m.}$ 52.^{s} temporis medii.

26. Denique celeberrimum *Minimi Crepuſculi* Problema nihil aliud eſt niſi purum putumque Problematis iſtius corollarium. In illo enim quaeritur anni dies, quo Sol velociſſime ſpatium trajicit comprehenſum ab horizonte, & circulo crepuſculari ſive almicantarath decem octo circiter gradibus infra horizontem depreſſo. Formula §. 18. $\textit{ſin. } x = \textit{ſin. } l \textit{ tang. } \frac{1}{2} a$ quaeſitam ſuppeditat Solis declinationem, adeoque anni tempus, quo crepuſculum omnium breviſſimum obſervatur. Quum autem hic ſumatur a negative, quippe quae depreſſionem exprimit circuli crepuſcularis; prodit icc irco $\textit{ſin. } x = - \textit{ſin. } l. \textit{ tang. } \frac{1}{2} a$. Quocirca in regionibus Borealibus Solis declinatio minimi Crepuſculi tempore eſt ſemper Auſtralis, hoc eſt minimum contingit crepuſculum Sole in ſignis auſtralibus verſante.

27. Minimi crepuſculi duratio habetur ex formula §. 19. $\textit{ſin. } \frac{1}{2} (h' - h) = \frac{\textit{ſin. } \frac{1}{2} a}{\textit{cos. } l}$, qua in formula quum quantitates a, & $h' - h$ in hac hypotheſi ſint utrinque

negativae, ſpectari ideo potuerunt veluti poſitivae, & $\mathit{ſin.}\ \frac{1}{2}(h' - h)$ permutari poterit cum $\mathit{ſin.}\ \frac{1}{2}(h - h')$. Quum autem ſit $\frac{1}{2}a = 9°$, ſi fuerit $l = 81°$, orietur $\mathit{ſin.}\ \frac{1}{2}(h - h') = 1$, hoc eſt $h - h' = 180°$: videlicet minimi Crepuſculi duratio in latitudine terreſtri 81° conſtat 12.$^{hor.}$ ſidereis, vel 11.$^{h.}$ 58.^{m} 2.$^{s.}$ ſolaribus mediis. In latitudine autem majori duratio elicitur abſurda, nimirum $\mathit{ſin.}\ \frac{1}{2}(h - h') > 1$. Duratio demum minimi Crepuſculi breviſſima, ſeu *Minimarum Minima* ſub aequatore manifeſtatur, & quidem aequinoctiorum tempore; nam fit $\mathit{ſin.}\,x = -\mathit{ſin.}\,l\ \mathit{tang.}\,\frac{1}{2}a$

$$= 0,\ \&\ \mathit{ſin.}\ \frac{1}{2}(h - h') = \frac{\mathit{ſin.}\ \frac{1}{2}a}{\mathit{cos.}\ l} = \mathit{ſin.}\ \frac{1}{2}a$$

$= \mathit{ſin.}\ 9°$, ſeu $h - h' = 18°$; quod praebet 1.$^{hor.}$ 12.$^{m.}$ temporis ſiderei, vel 1.$^{h.}$ 11.$^{m.}$ 47.$^{s.}$ temporis medii.

28. In hac noſtra Ticinenſi latitudine dies, & duratio minimi Crepuſculi ex formulis $\mathit{ſin.}\,x =$

$$-\mathit{ſin.}\,l\ \mathit{tang.}\,\frac{1}{2}a,\ \&\ \mathit{ſin.}\,\frac{1}{2}(h - h') = \frac{\mathit{ſin.}\ \frac{1}{2}a}{\mathit{cos.}\ l}$$ propere

inveniuntur:

l	45.° 11'
a	18.°
$\log.\ \mathit{ſin.}\ l$	9, 8508702

log. tang. $\frac{1}{2}$ *a*	. . . .	9, 1997125
log. ſin. l tang. $\frac{1}{2}$ *a*	. . . . . .	9, 0505827

Igitur $x = - 6.° 27'$; quod cadit intra dies octavam, & nonam Octobris, & tertiam quartamve Martii.

Pro duratione minimi Crepuſculi habetur

log. ſin. $\frac{1}{2}$ *a*	. . . .	9, 1943324
log. cos. l	. . . .	9, 8480909
log. $\frac{ſin. \frac{1}{2} a}{cos. l} = ſin. (h - h')$	. .	9, 3462415

Propterea $\frac{1}{2}(h - h') = 12.° 49'. 24''$, ſive $h - h' = 25.° 38'. 48''$; quod praebet 1.$^{h.}$ 42.^{m} 35.$^{s.}$ temporis ſiderei, ſeu 1.$^{h.}$ 41.$^{m.}$ 18.$^{s.}$ temporis ſolaris medii.



DISQUISITIO VI.

De Astronomiae Nauticae Theorematibus.

1. VUlgatiſſimus MAUPERTUISII Liber *De Nautica Aſtronomia* Problematum utilitate & praeſtantia, argumenti novitate, ſolutionum brevitate & elegantia, ac denique nativa quadam geometrica venuſtate, quae MAUPERTUISIUM ſcribentem nunquam deſtituit, omnium Geometrarum promeruit ſuffragia. Verum elegantiſſimus Auctor praejudicata opinione adverſus Trigonometriam Sphaericam occupatus ejus uſum ſibi penitus interdixit, eoque inconſiderantiae proceſſit, ut vix ſibi temperaverit quominus illam Geometris, atque ipſis, ſi diis placet, Aſtronomis inutilem ac ſupervacaneam pronunciaret. *Pour réſoudre* (inquit in operis praefatione) *les Problemès Aſtronomiques, on a d' ordinaire recours à une Science ſecondaire: on les reduit à des triangles tracés ſur la ſurface de la Sphère, que cette Science apprend à réſoudre. Je parle de la Trigonométrie Sphérique: elle offre d' abord de grandes facilités. On trouve*

ſes régles à la tête de pluſieurs Livres : & ſouvent on réſont des queſtions importantes de l' Aſtronomie par une application aveugle de ces régles. Par elles on eſt diſpenſé de pénêtrer dans la nature de la queſtion ; & par elles l' Aſtronome ſe croiroit diſpenſé d' être Géomètre, s' il pouvoit méconnoître la Science à la quelle elles doivent leur origine. J' admire l' Art des prémiers Géomètres qui nous ont donné la Trigonométrie Sphérique : mais je crois que les Eſprits géométriques préféreront, pour les Problèmes d' Aſtronomie, des ſolutions immédiates à celles qu' on emprunte d' une autre Science ; & aux quelles on ne parvient qu' en pratiquant des régles dont l' origine n' eſt guére préſente à l' eſprit, & dont l' application eſt ſouvent ambigue. J' ai voulu délivrer l' Aſtronomie du béſoin de cette Science ſecondaire ; & la faire dépendre immédiatement de l' analyſe dont toutes les Sciences Mathématiques dépendent. Quaſi vero Trigonometriam Sphaericam Analyſis excluderet, vel componi cum illa non poſſet : quaſi tam multorum Problematum Aſtronomicorum analyticae ſolutiones a Trigonometria Sphaerica non eſſent depromptae : quaſi denique ſummorum Geometrarum opera, potiſſimum Aſtronomica hujus Trigonometriae Dictatis non niterentur. Hinc non immerito MAUPERTUISIUM reprehendit PEZENAS (*a*) conſulte

(a) *Aſtronomie des Marins, Pref.*

inquiens: *Je ne vois pas trop ſur quels principes Mr. MAUPERTUIS nous dit dans ſa Préface que la Trigonométrie Sphérique eſt une Science ſecondaire dans la réſolution des Problèmes Aſtronomiques. Je ſoutiens au contraire, que c'eſt la Science la plus directe; puiſque les Problèmes Aſtronomiques ne ſont eux mêmes que des triangles Sphériques, dont on cherche les angles ou les côtés.*

2. Quum itaque ingenii exercendi gratia Aſtronomiae Nauticae Problemata ad Sphaericam Trigonometriam revocare, eorumque analyticas ſolutiones ex ejus Trigonometriae principiis derivare aggreſſus eſſem, tam ex voto res ceſſit, ut ex binis tantum ſimpliciſſimis Sphaericorum triangulorum proprietatibus utiliora quaeque ac praecipua Nauticae Aſtronomiae Theoremata praeter ſpem mihi inferre contigerit adhibitis ad hanc rem ſpeciebus ſinuum & coſinuum Eulerianis ad imaginationem & memoriam juvandam aptiſſimis, ablegatiſque ſymbolis Maupertuiſianis valde incommodis & moleſtis. Binae autem illae triangulorum ſphaericorum proprietates, ex quibus Nauticae Aſtronomiae partem maximam intuli, ad Lemmata duo ſequentia referuntur:

LEMMA I.

3. IN Triangulo quovis Sphaerico coſinus lateris unius cujuslibet aequatur facto ex ſinubus laterum reliquorum ducto in coſinum anguli ab hiſce comprehenſi, nec non facto ex coſinubus laterum eorundem.

LEMMA II.

4. IN Triangulo Sphaerico quocumque ſinus angulorum ſe habent ut ſinus laterum oppoſitorum.

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5. Sit jam in Triangulo Sphaerico *PZS*, *P* Polus, *Z* Loci Zenit, *S* Sol vel Aſtrum quodcumque. Hic ponimus, verſari Aſtrum in Hemisphaerio Poli elevati; levis quippe algebraici ſigni mutatio ipſum transferet quoties libuerit in Hemisphaerium Poli depreſſi.

Quamobrem arcus *PZ* erit complementum latitudinis Loci, arcus *ZS* complementum altitudinis Aſtri ſupra Horizontem, arcus *PS* complementum ejus declinationis; demum angulus *ZPS* erit angulus Horarius, & angulus *PZS* angulus Azimuthalis. Conſtat

autem, Aſtronomiam Nauticam fere univerſam in hiſce quinque niti elementis, Latitudine Loci, Altitudine Aſtri ſupra Horizontem, ejus Declinatione, Angulo Horario, Angulo Azimuthali.

6. Dicatur igitur.

Loci Latitudo	=	l
Aſtri ſupra Horizontem Altitudo . .	=	a
Aſtri Declinatio	=	d
Angulus Horarius	=	h
Angulus Azimuthalis	=	z

Hinc vero colligitur

$$ZPS = h$$

$$PZS = z$$

$$\text{ſin.}\ PZ = \cos.\ l$$

$$\text{ſin.}\ ZS = \cos.\ a$$

$$\text{ſin.}\ PS = \cos.\ d$$

$$\cos.\ PZ = \text{ſin.}\ l$$

$$\cos.\ ZS = \text{ſin.}\ a$$

$$\cos.\ PS = \text{ſin.}\ d$$

7. Ex quinque jam Elementis a, d, h, l, z quinque oriuntur quaterniones, nimirum

I.

a, d, h, l

II.

a, d, l, z

III.

a, d, h, z

IV.

d, h, l, z

V.

a, h, l, z

horumque ſinguli Theoremata quatuor praebent diverſa.

---

8. Sit igitur quaternio primus.

I.

a, d, h, l

Ex hoc ſequentia quatuor Theoremata deducuntur:

THEOREMA I.

$$Sin.\,a = cos.\,l\;cos.\,d\;cos.\,h + ſin.\,l.\,ſin\,d.$$

Dem. Per Lemma I. eſt $cos.\,ZS = ſin.\,PZ \times ſin.\,PS\;cos.\,ZPS + cos.\,PZ\;cos.\,PS$ Ergo &c. *Q.E.D.*

THEOREMA II.

$$Cos.\,h = \frac{ſin.\,a - ſin.\,l.\,ſin.\,d}{cos.\,l\;cos.\,d}.$$

Dem. Liquet ex ſimplici reductione formulae Theorematis I. *Q.E.D.*

9. ## THEOREMA III.

$$Sin.\,l = \frac{ſin.a\,ſin.d \pm cos.d\,cos.h\sqrt{(ſin.^2 d - ſin.^2 a + cos.^2 d\,cos.^2 h)}}{ſin.^2 d + cos.^2 d\,cos.^2 h}.$$

Dem. In formula primi Theorematis ſubrogatur $\sqrt{(1 - \text{ſin.}^2 l)}$ loco *cos. l*, & ſublata irrationalitate habetur aequatio quadratica, cujus radix eſt *ſin. l* = &c. *Q. E. D.*

10. # THEOREMA IV.

$$\text{Sin.}\,d = \frac{\text{ſin.}a\,\text{ſin.}l \pm \text{cos.}l\,\text{cos.}h\sqrt{(\text{ſin.}^2 l - \text{ſin.}^2 a + \text{cos.}^2 l\,\text{cos.}^2 h)}}{\text{ſin.}^2 l + \text{cos.}^2 l.\,\text{cos.}^2 h}.$$

Dem. Eodem modo ac Theorema III. *Q. E. D.*

11. Sit modo quaternio ſecundus

I I.

a, *d*, *l*, *z*

Inde eruuntur Theoremata quatuor ſequentia

THEOREMA V.

$$\text{Sin.}\ d = \text{cos.}\ l\ \text{cos.}\ a\ \text{cos.}\ z + \text{ſin.}\ l\ \text{ſin.}\ a.$$

Dem. Par eſt ratiocinatio ac Theorematis I. *Q. E. D.*

THEOREMA VI.

$$Cos.\ z = \frac{ſin.\ d - ſin.\ l\ ſin.\ a}{cos.\ l\ cos.\ a.}$$

Dem. Conſectarium eſt Theorematis V. *Q. E. D.*

12. THEOREMA VII.

$$Sin.\ l = \frac{ſin.\ d\ ſin.\ a \pm cos.\ a\ cos.\ z \sqrt{(ſin.^2 a - ſin.^2 d + cos.^2 a\ cos.^2 z)}}{ſin.^2\ a + cos.^2\ a\ cos.^2\ z}$$

Dem. Facta ſubſtitutione quantitatis $\sqrt{(1 - ſin.^2\ l)}$ pro *cos. l* in formula Theorematis V., ſublataque porro aſymmetria, oritur aequatio ſecundi gradus, cujus invenitur radix *ſin. l* = &c. *Q. E. D.*

13. THEOREMA VIII.

$$Sin.\ a = \frac{ſin.\ d\ ſin.\ l \pm cos.\ l\ cos.\ z \sqrt{(ſin.^2 l - ſin.^2 d + cos.^2 l\ cos.^2 z)}}{ſin.^2\ l + cos.^2\ l\ cos.^2\ z}$$

Dem. Eadem quae Theorematis VII. *Q. E. D.*

14. Sequitur quaternio tertius.

III.

a, d, h, z

Et ex hoc Theoremata alia quatuor.

THEOREMA IX.

$$Sin.\ z = \frac{ſin.\ h.\ cos.\ d}{cos.\ a}.$$

Dem. Per Lemma II. *ſin. z : ſin. h : : ſin. PS : ſin ZS : : cos. d : cos. a.* Ergo &c. *Q. E. D.*

15. THEOREMA X.

$$Sin.\ h = \frac{ſin.\ z\ cos.\ a}{cos.\ d}.$$

Dem. Ex praecedenti. *Q. E. D.*

16. ## THEOREMA XI.

$$Cos.\ d = \frac{ſin.\ z\ cos.\ a}{ſin.\ h}.$$

Dem. Uti prius. *Q. E. D.*

17. ## THEOREMA XII.

$$Cos.\ a = \frac{ſin.\ h.\ cos.\ d}{ſin.\ z}.$$

Dem. Ut ſupra. *Q. E. D.*

18. Pervenimus nunc ad quartum quaternionem.

I V.

d, h, l; z

unde quatuor alia Theoremata colliguntur.

THEOREMA XIII.

$$Sin.\,z = \frac{\text{ſin}.\,h.\,cos.\,d.}{\sqrt{(1 - (cos.l\,cos.d.\,cos.\,h + \text{ſin}.l\,\text{ſin}.d)^2)}}.$$

Dem. In formula Theorematis IX. ſubſtituitur pro *cos. a* ejus valor ex Theoremate I. deductus, atque inde oritur *ſin. z* = &c. *Q. E. D.*

19. THEOREMA XIV.

$$Cos.\,h = \frac{A \pm \sqrt{(A^2 + BC)}}{B},$$ facto ſcilicet

$A = \text{ſin}.\,l\; cos.\,l\; \text{ſin}.\,d\; cos.\,d\; \text{ſin}.^2\, z$

$B = cos.^2\, d - cos.^2\, d\; cos.^2\, l\; \text{ſin}.^2\, z$

$C = cos.^2\, d + \text{ſin}.^2\, l\; \text{ſin}.^2\, z\; \text{ſin}.^2\, d - \text{ſin}.^2\, z$

Dem. In formula Theorematis X. pro *ſin. h* ſubrogatur $\sqrt{(1 - cos.^2\, h)}$, & pro *cos. a* valor ipſius ex Theoremate I. Tum ſublata irrationalitate naſcitur quadratica aequatio, cujus radix eſt *cos. h* = &c. *Q. E. D.*

20. THEOREMA XV.

$$Cos.d = \sqrt{\left(\frac{D \pm \sqrt{(D^2 - FE)}}{F}\right)}, \text{ aſſumpto}$$

ſcilicet

$$D = 2 ſin.^4 z\, ſin.^2 l\, cos.^2 l\, cos.^2 h + ſin.^4 z\, cos.^4 l\, cos.^2 h$$
$$+ ſin.^2 z\, ſin.^2 h\, cos.^2 l - ſin.^4 z\, ſin.^2 l\, cos.^2 l$$
$$E = ſin.^4 z\, cos.^4 l$$
$$F = 4\, ſin.^4 z\, ſin.^2 l\, cos.^2 l\, cos.^2 h$$
$$+ (ſin.^2 h + ſin.^2 z\, cos.^2 l\, cos.^2 h - ſin.^2 z\, ſin.^2 l)^2.$$

Dem. In formula Theorematis XI. pro *cos. a* ponitur ipſius valor ex Theoremate I. eliciendus; tum aufertur asymmetria, & occurrit factum *cos. d ſin. d*, hoc eſt $cos. d \sqrt{(1 - cos.^2 d)}$: eliminatur iterum asymmetria, oriturque tandem aequatio *derivativa* quarti gradus, cujus radix eſt *cos. d* = &c. *Q. E. D.*

21. THEOREMA XVI.

$$ſin. l = \frac{\pm ſin. h\, cos. h\, cos. z\, cos.^2 d + ſin. d \sqrt{(ſin.^2 z - ſin.^2 h\, cos.^2 d)}}{ſin. z\, ſin.^2 d + ſin. z\, cos.^2 d\, cos.^2 h}.$$

Dem. In formula Theorematis III. ſubſtituitur pro *ſin. a*, & *ſin.*² *a* eorum valor ex Theoremate XII. eruendus. *Q. E. D.*

22. Sequitur poſtremo quintus quaternio.

V.

a, h, l, z

& ex hoc alia quatuor Theoremata.

THEOREMA XVII.

$$Sin.\, h = \frac{ſin.\, z\, cos.\, a}{\sqrt{[1-(cos.\, l\, cos.\, a\, cos.\, z + ſin.\, l\, ſin.\, a)^2]}}.$$

Dem. In formula Theorematis X. ponitur pro *cos. d* ejus valor ex Theoremate V. eliciendus. *Q. E. D.*

23. THEOREMA XVIII.

$$Cos.\, z = \frac{G \pm \sqrt{(G^2 + HI)}}{H},$$ aſſumpto ſcilicet

$G = \text{ſin.}\, l \cos.\, l\ \text{ſin.}\, a \cos.\, a\ \text{ſin.}^2 h$

$H = \cos.^2 a - \text{ſin.}^2 h \cos.^2 l \cos.^2 a$

$I = \text{ſin.}^2 h\ \text{ſin.}^2 l\ \text{ſin.}^2 a + \cos.^2 a - \text{ſin.}^2 h$

Dem. Eliminata irrationalitate a formula Theorematis praecedentis, poſitoque $1 - \cos.^2 z$ pro $\text{ſin.}^2 z$ obtinetur quadratica aequatio, quae radicem habet $\cos. z$ = &c. *Q.E.D.*

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24. THEOREMA XIX.

$$\cos. a = \sqrt{\left(\frac{K \pm \sqrt{(K^2 - LM)}}{L}\right)},$$ aſſumpto

videlicet

$K = \cos.^2 l \cos.^2 z\ \text{ſin.}^4 h + \cos.^2 l\ \text{ſin.}^2 z\ \text{ſin.}^2 h$
$- \text{ſin.}^2 z\ \text{ſin.}^2 l \cos.^2 l\ \text{ſin.}^4 h$

$L = 4\, \text{ſin.}^2 l \cos.^2 l \cos.^2 z\ \text{ſin.}^4 h$
$+ (\text{ſin.}^2 z + \cos.^2 z\ \text{ſin.}^2 h \cos.^2 l - \text{ſin.}^2 h\ \text{ſin.}^2 l)^2$

$M = \text{ſin.}^4 h \cos.^4 l$

Dem. Tollitur asymmetria in formula Theorematis XVI., tum ſubſtituitur $\sqrt{(1 - cos.^2 a)}$ loco $ſin. a$, ac tollitur iterum asymmetria. Inde oritur aequatio *derivativa* quarti gradus, cujus radix praebet $cos. a =$ &c. *Q.E.D.*

25. THEOREMA XX.

$$Sin. l = \frac{1}{P} \sqrt{\left(N \pm \sqrt{(N^2 - P^2 Q^2)}\right)},$$ poſito nimirum

$$N = 2\, ſin.^2 a\, cos.^2 a\, cos.^2 z\, ſin.^4 h$$
$$(ſin.^2 h\, ſin.^2 a - ſin.^2 h\, cos.^2 z\, cos.^2 a) \times$$
$$(ſin.^2 z\, cos.^2 a + ſin.^2 h\, cos.^2 z\, cos.^2 a - ſin.^2 h)$$
$$P = ſin.^2 h\, ſin.^2 a + ſin.^2 h\, cos.^2 z\, cos.^2 a$$
$$Q = ſin.^2 z\, cos.^2 a + ſin.^2 h\, cos.^2 z\, cos.^2 a - ſin.^2 h.$$

Dem. A formula Theorematis XVII. eliminatur irrationalitas, deinde loco $cos. l$ ſubrogatur ejus valor $\sqrt{(1 - ſin.^2 l)}$, ac rurſus tollitur irrationalitas. Hoc facto prodit aequatio *derivativa* quarti gradus, hujuſ-

que radix dat $ſin.\ l =$ &c. *Q.E.D.*

26. Evolutis Theorematis omnibus, quae ex variae quinque elementorum a, d, h, l, z combinatione derivantur, duodecim alia poſſunt ex praecedentibus colligi, eaque longe ſimpliciora ac fortaſſis utiliora, ſi ponamus verſari Aſtrum in Horizonte; ſeu nullam eſſe ipſius altitudinem, qua in hypotheſi horarius angulus h tempus indicat ab Aſtro inſumptum in aſcenſu ab Horizonte ad Meridianum, diciturque iccirco arcus ſemidiurnus, magni apud Aſtronomos uſus. Denotante jam igitur h arcum hujuſmodi ſemidiurnum, factaque $a = 0$, ex praecedentibus quaternionibus quatuor iſti

$$a,\ d,\ h,\ l$$

$$a,\ d,\ l,\ z$$

$$a,\ d,\ h,\ z$$

$$a,\ h,\ l,\ z$$

degenerant in terniones

I.

d, *h*, *l*

II.

d, *l*, *z*

III.

d, *h*, *z*

IV.

h, *l*, *z*

Quorum ſinguli tria praebent Theoremata. Sit igitur ternio primus.

I.

d, *h*, *l*

Ex hoc tria ſequentia Theoremata derivantur.

THEOREMA XXIX. &c.

THEOREMA XXI.

$$Cos. h = - tang. l\ tang. d.$$

Dem. In hac hypotheſi formula Theorematis I. mutatur in $cos. l\ cos. d\ cos. h + ſin. l\ ſin. d = 0$. Hinc fit

$$cos. h = - \frac{ſin. l\ ſin.\ d}{cos. l\ cos. d} = - tang. l\ tang. d.\ Q.E.D.$$

27. # THEOREMA XXII.

$$Tang. l = - \frac{cos.\ h}{tang. d}.$$

Dem. Patet ex praecedenti. *Q.E.D.*

28. # THEOREMA XXIII.

$$Tang. d = - \frac{cos. h}{tang. l}.$$

Dem. Eadem. *Q.E.D.*

29. Sequitur ternio ſecundus.

I I.

d, l, z

Huic reſpondent tria iſthaec Theoremata:

THEOREMATA XXIV.

$$Cos.\,z = \frac{ſin.\,d}{cos.\,l}.$$

Dem. Formula Theorematis VI. in hanc mutatur, facto *ſin. a* = 0, & *cos. a* = 1. *Q.E.D.*

30. # THEOREMA XXV.

$$Sin.\,d = cos.\,l\;cos.\,z$$

Dem. liquet ex praecedenti. *Q.E.D.*

31. ## THEOREMA XXVI.

$$Cos. l = \frac{ſin. d}{cos. z}.$$

Dem. Eadem. *Q.E.D.*

32. Ternio tertius

I I I.

d, *h*, *z*

tria haec parit Theoremata:

THEOREMA XXVII.

$$Sin. z = ſin. h \; cos. d.$$

Dem. Patet ex formula Theorematis IX. *Q.E.D.*

33. THEOREMA XXVIII.

$$\text{ſin}.\,h = \frac{\text{ſin}.\,z}{\cos.\,d}.$$

Dem. Ex praecedenti. *Q. E. D.*

34. THEOREMA XXIX.

$$\text{Cos}.\,d = \frac{\text{ſin}.\,z}{\text{ſin}.\,h}.$$

Dem. Ut ſupra. *Q. E. D.*

35. Denique ex Ternione quarto

I V.

h, *l*, *z*

alia haec tria deducuntur Theoremata:

THEOREMA XXX.

$$Tang.\, h = \frac{tang.\, z}{ſin.\, l}.$$

Dem. Ex transformatione formulae Theorematis XVII. oritur

$$ſin.\, h = \frac{ſin.\, z}{\sqrt{(1 - cos.^2 l\; cos.^2 z)}}.$$ Ergo &c. *Q.E.D.*

36. THEOREMA XXXI.

$$Sin.\, l = \frac{tang.\, z}{tang.\, h}.$$

Dem. Patet ex reductione formulae praecedentis. *Q.E.D.*

THEOREMA XXXII.

Tang. z = *ſin.* l *tang.* h.

Dem. Liquet ex praecedenti. *Q.E.D.*



DISQUISITIO VII.

De Cometarum Motu.

1. Poſtquam diurnae Parallaxeos defectus, vel minima prorſus magnitudo Cometas extulit ſupra regiones ſublunares, annua vero Parallaxis oſtendit eorundem in regiones Planetarum deſcenſum, illud ſtatim ex Newtonianae Philoſophiae dictatis occurrit & communi Aſtronomorum ſuffragio firmatum eſt, Cometas omnes Planetarum inſtar ferri in orbibus ellipticis circa Solem in communi orbium umbilico poſitum, urgentemque momentis ſingulis Planetas omnes & Cometas praepotenti attractionis vi, quae in ratione duplicata reciproca diſtantiarum a Sole augeatur. Verum ea intercedit inter orbes Planetarum & Cometarum diſſimilitudo, ut illi ellipſes quidem ſint, ſed ad circulos proxime accedant, iſti vero ellipſes ſint acutiſſimae ac multum excentricae, habentes nimirum conjugatorum axium alterum longiſſimum, alterum breviſſimum. Diſcrimen hoc a projectionis velocitate proficiſci docet Newtoniana Philoſophia: quum enim ad deſcriptionem circuli ea poſtuletur projectionis velocitas,

quae requiritur ut corpus ad duplam altitudinem ſeu diſtantiam a centro poſſit aſcendere, & ad deſcriptionem ellipſeos opus ſit, ut projectionis velocitas in quavis altitudine ſe habeat ad velocitatem in circulo in eadem altitudine in minori ratione quam radix binarii ad unitatem, ad deſcribendam vero parabolam celeritatem projectionis in quavis altitudine talem eſſe oporteat, quae ad celeritatem in circulo in eadem altitudine rationem habeat omnino eandem quam binarii radix ad unitatem; inde conſequitur, Planetas ab initio projectos fuiſſe impetu parum ab illo diverſo quocum ad duplam altitudinem potuiſſent pervenire, Cometas vero eam accepiſſe projectilem celeritatem, quae ad celeritatem in circulo in pari altitudine rationem habeat vix minorem, quam radix numeri binarii ad unitatem. Quo ſane factum eſt, ut elliptici Planetarum orbes ferme circulares evaderent, orbeſque Cometarum in Solis vicinia vix a parabolicis differrent. Quum itaque Cometae cujuslibet ſemita ſaltem prope Solem, ſive apſidem imam ſit ad ſenſum parabolica, commodiſſima hinc patuit Aſtronomis via in Cometarum motibus ſupputandis, ut ſcilicet neglecta ellipſi, quae ob inextricabiles calculos non aliter eſſet quam herculeo labore tractabilis, parabolam excolerent, cujus tetragoniſmus jam inde ab Archimedis aetate Geometris

perſpectus & obvius Cometarum loca ad tempus datum labore non ita immani, nec prorſus ingrato ſuppeditat.

2. Jam vero in definiendis Cometae elementis illud occurrit praecipuum, ut cognito angulo, quem rectae a duobus Cometae locis ad centrum Solis ductae comprehendunt, cognitaque earundem rectarum, ſeu radiorum vectorum magnitudine quamproxime, inveniatur diſtantia Cometae perihelia, hoc eſt diſtantia foci a parabolae vertice, nec non poſitio axis, ipſaque iccirco parabola delineari poſſit, & graphice repraeſentari. Ad hoc praeſtandum expedita mihi ſe obtulit methodus, quae cum differat ab aliis, & ſimplicitate atque elegantia ſe commendet, aequiorum Geometrarum non detrectat judicium.

3. Sit itaque *CPM* orbis parabolici pars in Solis vicinia, & ex binis Cometae locis *C*, *M* jungantur Fig. 5. ad Solis centrum *S* in orbis umbilico conſtitutum radii vectores *CS*, *SM* datum angulum *CSM* comprehendentes, notaque praeterea ſit radiorum *CS*, *SM* magnitudo ſaltem vero proxima: quaeritur diſtantia perihelia *SP*, & poſitio axis *QP*. Fiat

$$CS = a,$$
$$MS = b,$$
$$SP = p,$$
$$\text{angulus } CSM = \varphi,$$

$$CSP = x,$$
$$PSM = \varphi - x.$$

Ductis ordinatis CD, ME habetur $SE = b\ cos.(\varphi - x)$, $SD = a\ cos.x$, $ME^2 = 4p.PE = 4p.(SP - SE)$ $= 4p^2 - 4pb\ cos.(\varphi - x)$, $CD^2 = 4p.DP = 4p.(SP - SD) = 4p^2 - 4pa\ cos.x$. Igitur $SM^2 = b^2$ $= 4p^2 - 4pb\ cos.(\varphi - x) + b^2\ cos.^2(\varphi - x)$, & $SC^2 = a^2 = 4p^2 - 4pa\ cos.x + a^2\ cos.^2 x$; eductaque radice prodibit $b = 2p - b\ cos.(\varphi - x)$, $a = 2p$ $- a\ cos.x$. Dempta harum aequationum prima ab altera oritur $b\ cos.(\varphi - x) - a\ cos.x = a - b$, hoc eſt $(b\ cos.\varphi - a)\ cos.x + b\ ſin.\varphi\ ſin.x = a - b$, vel $(b\ cos.\varphi - a)\ cos.x + b - a = - b\ ſin.\varphi \sqrt{(1 - cos.^2 x)}$, & quadrando $(b\ cos.\varphi - a)^2\ cos.^2 x + 2(b - a) \times$ $(b\ cos.\varphi - a)\ cos.x + (b - a)^2 = b^2\ ſin.^2\varphi$ $- b^2\ ſin.^2\varphi\ cos.^2 x$, & factis reductionibus, expurgataque de more aequatione invenitur $cos.^2 x$

$$+ \frac{2(b-a)(b\ cos.\varphi - a)\ cos.x}{b^2 - 2ab\ cos.\varphi + a^2} + \frac{b^2\ cos.^2\varphi - 2ab + a^2}{b^2 - 2ab\ cos.\varphi + a^2} = 0.$$

Igitur $cos.x = \frac{(a-b)(b\ cos.\varphi - a)}{b^2 - 2ab\ cos.\varphi + a^2} \pm \sqrt{\left(\frac{A^2}{B^2} - \frac{C}{B}\right)}$,

facta ſcilicet

$$A = (a - b)(b\ cos.\varphi - a),$$
$$B = b^2 - 2ab\ cos.\varphi + a^2, \&$$
$$C = b^2\ cos.^2\varphi - 2ab + a^2.$$

Subducto porro calculo deprehenditur $\frac{[illegible]}{B^2}$ —

$$\mp \frac{A^2 - CB}{B^2} = \frac{2ab^3(1 - \cos.\varphi\,\text{ſin.}^2\varphi - \cos.^2\varphi)}{b^2 - 2ab\cos.\varphi + a^2};$$

proindeque ſupereſt $\cos.\,x$

$$= \frac{(a-b)(b\cos.\varphi - a) \pm \sqrt{2ab^3(1 - \cos.\varphi\,\text{ſin.}^2\varphi - \cos.^2\varphi)}}{b^2 - 2ab\cos.\varphi + a^2};$$

quae ſane expreſſio quaeſiti anguli x, ſeu CSP quantitatem exhibet, & conſequenter axis PQ poſitionem definit. Diſtantia autem perihelia p, ſeu quarta parametri pars obtinetur ex ſuperiori aequalitate $a = 2p - a\cos.\,x$, habeturque $p = \tfrac{1}{2}a(1 + \cos.\,x)$, & poſt idoneas reductiones p

$$= \frac{\tfrac{1}{2}a(ab + b^2)(1 - \cos.\varphi) \pm \tfrac{1}{2}a\sqrt{2ab^3(1 - \cos.\varphi\,\text{ſin.}^2\varphi - \cos.^2\varphi)}}{b^2 - 2ab\cos.\varphi + a^2}$$

$$= \frac{a(ab + b^2)\,\text{ſin.}^2\tfrac{1}{2}\varphi \pm \tfrac{1}{2}a\sqrt{2ab^3(1 - \cos.\varphi\,\text{ſin.}^2\varphi - \cos.^2\varphi)}}{b^2 - 2ab\cos.\varphi + a^2}.$$

4. Deſignet jam unitas, ut exemplo res illuſtretur, diſtantiam Terrae mediocrem a Sole, ſitque porro

b	$= 4,6893$
a	$= 4,5634$
$a - b$	$= -0,1259$

φ	$= 87°$
log. b	$= 0,6711080$
log. a	$= 0,659288$
log. cos. φ	$= 8,7188002$
log. b cos. φ	$= 9,3899082$
b cos. φ	$= 0,2454$
b. cos. $\varphi - a$	$= -4,3180$
log. $-(b\ cos.\ \varphi - a)$	$= 0,6352826$
log. $-(a - b)$	$= 9,1000257$
log. $(a-b)(b\,cos.\,\varphi - a)$ . . .	$= 9,7353083$
$(a-b)(b\,cos.\,\varphi - a)$	$= 0,5436$
log. ſin. φ	$= 9,9994044$
log. ſin.^{2} φ	$= 9,9988088$
log. cos. φ	$= 8,7188002$
log. cos. φ *ſin.*^{2} φ	$= 8,7176090$
cos. φ *sin.*^{2} φ	$= 0,05219$
log. cos.^{2} φ	$= 7,4376004$
cos.^{2} φ	$= 0,002739$
cos. φ *sin.*^{2} φ + *cos.*^{2} φ	$= 0,05493$

$1 - \cos.\ \varphi\ \sin.^2\ \varphi - \cos.^2\ \varphi \ldots = 0,\ 945$[illegible]$7$

$log.\ 2 \ldots\ldots\ldots = 0,\ 3010300$

$log.\ a \ldots\ldots\ldots = 0,\ 6592885$

$log.\ b^3 \ldots\ldots\ldots = 2,\ 0133240$

$log.\ 2ab^3 \ldots\ldots\ldots = 2,\ 9736425$

$log.\ (1 - \cos.\ \varphi\ \sin.^2\ \varphi - \cos.^2\ \varphi) = 9,\ 9754640$

$log.\ 2ab^3\ (1 - \cos.\ \varphi\ \mathit{ſ}in.^2\ \varphi - \cos.^2\ \varphi) = 2,\ 9491065$

$log. \sqrt{2ab^3\ (1 - \cos.\ \varphi\ \sin.^2\ \varphi - \cos.^2 \varphi)} = 1,\ 4745532$

$\sqrt{2ab^3\ (1 - \cos.\ \varphi\ \sin.^2\ \varphi - \cos.^2\ \varphi)} = 29,\ 82$

$(a - b)\ (b \cos.\ \varphi - a) + \sqrt{[2ab^3 \times}$

$(1 - \cos.\ \varphi\ \mathit{ſ}in.^2\ \varphi - \cos.^2\ \varphi)] \ldots = 30,\ 3636$

$(a - b)\ (b \cos.\ \phi - a) - \sqrt{[2ab^3 \times}$

$(1 - \cos.\ \varphi\ \sin.^2\ \varphi - \cos.^2\ \varphi)] \ldots = -29,\ 2764$

$log. [(a - b)(b \cos.\ \varphi - a) + \sqrt{[2ab^3 \times}$

$(1 - \cos.\ \varphi\ \sin.^2\ \varphi - \cos.^2\ \varphi)]] \ldots = 1,\ 4823590$

$log.\ b^2 \ldots\ldots\ldots = 1,\ 3422160$

$log.\ a^2 \ldots\ldots\ldots = 1,\ 3185770$

$b^2 \ldots\ldots\ldots = 21,\ 99$

$a^2 \ldots\ldots\ldots = 20,\ 82$

$a^2 + b^2$ . . . . $= 42, 81$

$\log.\ b\ \cos.\ \varphi$ . . . . $= 9, 3899082$

$\log.\ a$ . . . . $= 0, 6592885$

$\log.\ 2$ . . . . $= 0, 3010300$

$\log.\ 2ab\ \cos.\ \varphi$ . . . . $= 0, 3502267$

$2ab\ \cos.\ \varphi$ . . . . $= 2, 24$

$a^2 + b^2 - 2ab\ \cos.\ \varphi$ . . . . . $= 40, 57$

$\log.(a^2 + b^2 - 2ab\ \cos.\ \varphi)$ . . . . $= 1, 6082050$

$$\log.\ \frac{(a-b)(b\ \cos.\varphi - a) + \sqrt{2ab^3(1-\cos.\varphi)\text{ſin.}^2\varphi - \cos.^2\varphi)}}{(a^2 + b^2 - 2ab\ \cos.\ \varphi)}$$

$=$ . . . . $= 9, 8741540$

Igitur angulus x, ſeu $CSP = 41°.\ 32'.\ 42''$, & $PSM = 45°.\ 27'.\ 18''$.

5. Inventis porro angulis x, & $\varphi - x$, hoc eſt veris Cometae anomaliis, ſtatim invenitur diſtantia Cometae perihelia p, ſeu PS ex formula $p = \frac{1}{2} a \times (1 + \cos.x)$:

$\cos.\ x$ . . . . $0, 7484351$

$\cos.\ x + 1$ . . . . $1, 7484351$

$\log.\ (\cos.\ x + 1)$ . . . . $0, 2426495$

$\log.\ \frac{1}{2}\ a$ . . . . $0, 3582585$

log. $\frac{1}{2}a(\cos. x + 1)$	. . . .	0, 6009080
$\frac{1}{2}a(\cos. x + 1)$	. . . .	3, 9894

Eſt ergo diſtantia perihelia quaeſita 3, 9894, hoc eſt fere quadrupla diſtantiae Telluris a Sole.

6. Si in expreſſionem *cos. x* §. 5. introducatur ſinus verſus anguli φ, nimirum ver. φ, ea valde contrahitur, ac formam ſimpliciorem adipiſcitur: quum enim ſit $1 - \cos.\varphi\, \mathit{ſin.}^2 \varphi - \cos.^2 \varphi = 1 - \cos.\varphi + \cos.^3 \varphi - \cos.^2 \varphi = 1 - \cos.\varphi - \cos.^2 \varphi\,(1 - \cos.\varphi) = \mathit{ver.}\,\varphi - \mathit{ver.}\,\varphi \cos.^2 \varphi = \mathit{ver.}\,\varphi\, \mathit{ſin.}^2 \varphi$, ſit ideo

$$\cos. x = \frac{(a-b)(b\cos.\varphi - a) \pm \sqrt{2ab^3(1-\cos.\varphi\, \mathit{ſin.}^2\varphi - \cos.^2\varphi)}}{b^2 - 2ab\cos.\varphi + a^2}$$

$$= \frac{(a-b)(b\cos.\varphi - a) \pm b\, \mathit{ſin.}\varphi \sqrt{2ab\, \mathit{ver.}\varphi}}{b^2 - 2ab\cos.\varphi + a^2}$$

7. Detegitur eodem artificio diſtantia perihelia p

$$= \frac{a(ab + b^2)\, \mathit{ſin.}^2 \tfrac{1}{2}\varphi \pm \tfrac{1}{2} ab\, \mathit{ſin.}\varphi \sqrt{2ab\, \mathit{ver.}\varphi}}{b^2 - 2ab\cos.\varphi + a^2},$$ quae

ſupputari haud moleſte poteſt etiam ante initam ſupputationem anguli x. Fit nempe

log. a	. . . .	0, 6592885
log. b	. . . .	0, 6711080
log. ab	. . . .	1, 3303965

ab	21, 3991
log. b^2	1, 3422160
b^2	21, 98953
$ab + b^2$	43, 38863
log. $(ab + b^2)$	1, 6373759
log. ſin. $\frac{1}{2}\varphi$	9, 8378122
log. ſin.^{2} $\frac{1}{2}\varphi$	9, 6756244
log. $a(ab + b^2)$ ſin.^{2} $\frac{1}{2}\varphi$	1, 9722888
$a(ab + b^2)$ ſin.^{2} $\frac{1}{2}\varphi$	93, 8185
log. $2a$	0, 9603185
log. b	0, 6711080
log. ver. φ	9, 9766544
log. $2ab$ ver. φ	1, 6080809
log. $\sqrt{2ab \text{ ver. } \varphi}$	0, 8040404
log. ſin. φ	9, 9994044
log. $\frac{1}{2}ab$	1, 0293665
log. $\frac{1}{2}ab$ ſin. $\varphi \sqrt{2ab \text{ ver. } \varphi}$	1, 8328113
$\frac{1}{2}ab$ ſin. $\varphi \sqrt{2ab \text{ ver. } \varphi}$	68, 0473
$a(ab+b^2)$ ſin.^{2} $\frac{1}{2}\varphi + \frac{1}{2}ab$ ſin. $\varphi\sqrt{2ab \text{ ver. } \varphi}$	161, 8658

$log. (a (ab + b^2) ſin.^2 \frac{1}{2} \varphi + \frac{1}{2} ab\, ſin. \varphi \times$

$\sqrt{2ab\, ver. \varphi})$ 2, 2091550

$log. (a^2 + b^2 - 2ab \cos. \varphi)$ . . . 1, 6082050

$$log. \frac{a\, ſin.^2 \frac{1}{2} \varphi (ab + b^2) + \frac{1}{2} ab\, ſin. \varphi \sqrt{2ab\, ver. \varphi}}{(a^2 + b^2 - 2ab \cos. \varphi)}$$

$=$ 0, 6009500

Igitur $p = 3, 9898$, quod non differt niſi 0, 0004 a praecedenti.

8. Definitis jam verarum anomaliarum angulis & diſtantia Cometae perihelia, haud difficulter defini- ri poterit tempus, quod in percurrendo arcu MC a da- tis radiis vectoribus MS, SC comprehenſo Cometa ab- ſumit. Fiat enimvero anomalia vera $CSP = \lambda$, altera $PSM = \omega$. Hinc nanciſcimur

$$CD = a\, ſin. \lambda \qquad ME = b\, ſin. \omega$$

$$SD = a \cos. \lambda \qquad SE = b \cos. \omega$$

$$PD = \frac{a^2 ſin.^2 \lambda}{4p} \qquad PE = \frac{b^2 ſin.^2 \omega}{4p}$$

Eſt autem Spatium Parabolicum $PME = \frac{b^3\, ſin.^3 \omega}{6p}$,

& $CPD = \frac{a^3\, ſin^3 \lambda}{6p}$: rurſus triangulum SEM

$= \frac{b^2 \sin.\omega \cos.\omega}{2}$, & triangulum $CSD = \frac{a^2 \sin.\lambda \cos.\lambda}{2}$.

Igitur erit Sector Parabolicus $CPMS$

$$= \frac{a^3 \sin.^3\lambda + 3p\, a^2 \sin.\lambda \cos.\lambda + b^3 \sin.^3 \omega + 3p\, b^2 \sin.\omega \cos.\omega}{6p}$$

9. Conſtat jam ex celeberrima Kepleriana Lege, tempora quibus diverſi Planetae vel Cometae arcus quoslibet orbitarum ſuarum percurrunt rationem ſequi ſimplicem directam arearum ab ipſis arcubus & binis radiis vectoribus interceptarum, & ſubduplicatam inverſam parametrorum; itaque ſi dicatur T tempus periodicum Terrae, unius nempe anni ſiderei, A area totius orbitae terreſtris, h ejus parameter, t vero tempus, quo Cometa deſcribit arcum MC, habebitur haec analogia

$$\frac{A}{\sqrt{h}} : \frac{a^3 \sin.^3\lambda + 3pa^2 \sin.\lambda \cos.\lambda + b^3 \sin.^3\omega + 3pb^2 \sin.\omega \cos.\omega}{12 \sqrt{p^3}} ::$$

$T : t$. Quare ſit t

$$= \frac{T(a^3 \sin.^3\lambda + 3pa^2 \sin.\lambda \cos.\lambda + b^3 \sin.^3\omega + 3pb^2 \sin.\omega \cos.\omega)}{12 A \sqrt{p^3}} \times$$

$\sqrt{h}$. Invenitur porro area A orbitae terreſtris ex nota ratione, quam habet ad aream circuli ſemiaxe tranſverſo orbitae tanquam radio deſcripti, quae ſcilicet

eadem eſt ac ratio ſemiaxis conjugati ad ſemiaxem tranſverſum. Quum enim ſemiaxem ellipſis terreſtris tranſverſum, ſeu diſtantiam terrae mediocrem a Sole § 4. unitati aequalem ſumpſerimus, denotante $1 : \pi$ rationem diametri ad peripheriam oritur circuli radio 1 deſcripti area $= \pi$; quamobrem ob $\sqrt{\frac{1}{2} h} =$ ſemiaxi conjugato orbitae terreſtris prodit $1 : \sqrt{\frac{1}{2} h} :: \pi : A$, hoc eſt $A = \pi \sqrt{\frac{1}{2} h}$. Inde poſtremo conſequitur t

$$= \frac{T(a^3 ſin.^3 \lambda + 3pa^2 ſin.\lambda cos.\lambda + b^3 ſin.^3 \omega + 3pb^2 ſin.\omega cos.\omega)}{12\pi \sqrt{\frac{1}{2} p^3}}$$

10. Jamvero exploratum eſt, tempus periodicum Terrae ad Fixas relatum complecti $365^d. 6^h. 9'. 11''$, nimirum dies 365, 2563773; itaque calculus inſtituetur hoc pacto.

$log.\ a$	0, 6592885
$log.\ ſin.\ \lambda$	9, 8216498
$log.\ a^3$	1, 9778655
$log.\ ſin.^3 \lambda$	9, 4649494
$log.\ a^3.\ ſin.^3 \lambda$	1, 4428149
$a^3\ ſin.^3 \lambda$	27, 72138
$log.\ 3$	0, 4771213
$log.\ p$	0, 6009500

log. a^2	1, 3185770
log. ſin. λ	9, 8216498
log. cos. λ	9, 8741541
log. $3pa^2$ ſin. λ cos. λ	2, 0924522
$3pa^2$ ſin. λ cos. λ	123, 7235
log. b	0, 6711080
log. ſin. ω	9, 8529066
log. b^3	2, 0133240
log. ſin.^{3} ω	9, 5587198
log. b^3 ſin.^{3} ω	1, 5720438
b^3 ſin.^{3} ω	37, 32878
log. 3	0, 4771213
log. p	0, 6009500
log. b^2	1, 3422160
log. ſin. ω	9, 8529066
log. cos. ω	9, 8460086
log. $3pb^2$ ſin. ω cos. ω	2, 1192025
$3pb^2$ ſin. ω cos. ω	131, 5838

a^3 ſin.^{3} $x + 3pa^2$ ſin. λ cos. $\lambda + b^3$ ſin.^{3} ω

$+ 3pb^2$ ſin. ω cos. $\omega = P$ 320, 35746

log. P	2, 5056348
log. T	2, 5625978
log. TP	5, 0682326
log. p	0, 6009500
log. p^3	1, 8028500
log. 72	1, 8573325
log. $72\,p^3$	3, 6601825
log. $\sqrt{72p^3}$	1, 8300912
log. π	0, 4971499
log. $\pi\sqrt{72\,p^3}$	2, 3272411
log. $\dfrac{TP}{\pi\sqrt{72p^3}}$	2, 7409915
$\dfrac{TP}{\pi\sqrt{72p^3}}$	550, 79683

Igitur Cometa in percurrendo arcu *MPC* tempus abſumit dierum quingentorum quinquaginta, horarum undeviginti.

11. Sagaciſſimus Geometra Lambertus, cujus immaturam mortem, dum haec ſcribo, mihi nuntiatam nunquam ſatis Diſciplinae Mathematicae lugebunt, in eximio Opuſculo de *Inſignioribus Orbitae Cometarum*

Proprietatibus, quo nihil legi poteſt elegantius, nihil ingenioſius, Problemate XV. inveniendum ſibi proponit tempus, quo Cometa deſcribit arcum *MPC* data ſumma radiorum vectorum *SC*, *SM* una cum chorda arcum *MPC* ſubtendente: facta vero eadem chorda $= k$ expreſſionem temporis nanciſcitur ſimplicitate & elegantia prope admirabilem, ſcilicet

$$T = \frac{\left(\frac{a+b+k}{2}\right)^{\frac{3}{2}} - \left(\frac{a+b-k}{2}\right)^{\frac{3}{2}}}{3m\sqrt{2}}$$, in qua valor

litterae m Lamberto eſt $\frac{1}{116,2648}$. Ut formulam hanc in exemplo praecedenti experiamur, ejuſque conſenſum cum formula noſtra oſtendamus, inveſtigetur prius valor chordae k, quae nimirum ex Geometria invenitur $= \sqrt{(a^2 + b^2 - 2ab\cos.\varphi)}$, hoc eſt §. 4. $k = \sqrt{40,57} = 6,36946$. Fiet igitur

a	4, 5634
b	4, 6895
k	6, 36946
$a+b+k$	15, 62216
$\frac{a+b+k}{2}$	7, 81108

$\log. \frac{a+b+k}{2}$	0, 8927110
$\log. \left(\frac{a+b+k}{2}\right)^3$	2, 6781330
$\log. \left(\frac{a+b+k}{2}\right)^{\frac{3}{2}}$	1, 3390665
$\left(\frac{a+b+k}{2}\right)^{\frac{3}{2}}$	21, 83065
$a+b$	9, 2527
$-k$	—6, 36946
$a+b-k$	2, 88324
$\frac{a+b-k}{2}$	1, 44162
$\log. \frac{a+b-k}{2}$	0, 1588508
$\log. \left(\frac{a+b-k}{2}\right)^3$	0, 4765524
$\log. \left(\frac{a+b-k}{2}\right)^{\frac{3}{2}}$	0, 2382762
$\left(\frac{a+b-k}{2}\right)^{\frac{3}{2}}$	1, 73092

$$\left(\frac{a+b+k}{2}\right)^{\frac{3}{2}}-\left(\frac{a+b-k}{2}\right)^{\frac{3}{2}} \ldots\ldots \quad 20, 09973$$

$$log.\left(\left(\frac{a+b+k}{2}\right)^{\frac{3}{2}}-\left(\frac{a+b-k}{2}\right)^{\frac{3}{2}}\right) \ldots \quad 1, 3031903$$

$$log.\frac{1}{m} \ldots\ldots\ldots\ldots \quad 2, 0654481$$

$$log.\frac{1}{m}\left(\left(\frac{a+b+k}{2}\right)^{\frac{3}{2}}-\left(\frac{a+b-k}{2}\right)^{\frac{3}{2}}\right) \ldots \quad 3, 3686384$$

$$log.\ 2 \ldots\ldots\ldots\ldots \quad 0, 3010360$$

$$log.\ \sqrt{2} \ldots\ldots\ldots\ldots \quad 0, 1505150$$

$$log.\ 3 \ldots\ldots\ldots\ldots \quad 0, 4771213$$

$$log.\ 3\sqrt{2} \ldots\ldots\ldots\ldots \quad 0, 6276363$$

$$log.\ \frac{1}{m}\,\frac{\left(\left(\frac{a+b+k}{2}\right)^{\frac{3}{2}}-\left(\frac{a+b-k}{2}\right)^{\frac{3}{2}}\right)}{3\sqrt{2}} \ldots \quad 2, 7410021$$

Quare oritur $T = 550, 8104$, quod ne ſemihora quidem differt a tempore ex formula noſtra prodeunte.

12. Si tempus quaeratur totius revolutionis Cometae per orbem ſuum, tempus ſcilicet periodicum, illud nequaquam poteſt per parabolam determinari: quandoquidem obſervationes illae, quae in Cometa prope Solem verſante nobiſque conſpicuo poſſunt inſtitui, ideo periodicum Cometae tempus exhiberent quod ex

illis majoris Ellipſeos axis longitudo colligeretur. Id vero nequit, ut per ſe liquet, per parabolam obtineri. Res autem eſſet infiniti prope operis immenſique laboris, Cometae orbem tamquam Ellipſim, qualis revera eſt, ſpectare, ejus Ellipſis formam ex obſervationibus definire, atque porro ex nota diſtantia umbilici a proximo vertice axis majoris longitudinem deducere. Praeterea tam immanis labor incaſſum magna ex parte ſuſciperetur parumque admodum accurationis inde ſperare fas eſſet, ſiquidem vel minimus in obſervando, iſque inevitabilis admiſſus error Elliptici orbis formam, indeque concluſam axis majoris menſuram maximopere immutat. Quum enim, ut notum eſt, minima ſufficiat variatio ad hoc, ut Ellipſis in Parabolam convertatur, & Ellipſeos axis infinitam longitudinem adipiſcatur; propterea aberratio multo adhuc minor poterit axem ipſum ſi non infinite, plurimum certe protendere vel contrahere.

13. Itaque longe conſultius erit ex orbis parabolici poſitione, ex parametri longitudine, ex minima Cometae a Sole diſtantia Cometam unum ab altero internoſcere, ita ut ſi bini Cometarum orbes in hiſce notis & caracteribus ſecum invicem undequaque conſentiant, ac parva, ſi quae occurrit, diſcrepantia vel obſervationibus non omnimode accuratis, vel mutatio-

nibus in Cometae curſu incidentibus adſcribi jure poſſit, orbes ipſi ſpecie tenus diverſi pro uno eodemque habendi ſint, in quo unicus Cometa revolvatur. Si praeterea tempora etiam, poſt quorum lapſum Cometa ad perihelium regreditur, inveniantur aequalia, nulla ſane ſupererit de Cometae identitate incertitudo aut haeſitatio, ſimulque tempus ipſius periodicum tuto inde certiſſimeque colligetur.

14. Talis orbium congruentia detecta potiſſimum fuit in Cometa, qui anno 1456. die 9. Junii, anno 1531. die 28. Februarii, anno 1607. die 26. Octobris, & anno 1682. die 14. Septembris ad Solem maxime appropinquavit. Tempus inter binas proximas hujus Cometae apparitiones elapſum complectitur annos circiter 75½. Quapropter ejus Cometae reditus jure merito expectabatur anno 1759., quemadmodum ipſo contigit, quum die 13. Martii ipſe perihelium attigerit. Sed de hujus Cometae motu, periodo, perturbatione incredibili ingenii ſagacitate & induſtria egit ſummus Geometra Clairaut, qui eximio edito opere *Théorie du Mouvement des Cometes* immani exantlato calculorum labore omnes illius Cometae perturbationes, indeque ortam periodici temporis protractionem definivit.

15. Ex noto iſtius Cometae tempore periodico T inveniri facile poteſt media ipſius a Sole diſtantia D,

ſeu ſemiaxis major Ellipſeos per exploratiſſimam Aſtronomiae legem, juxta quam periodica Planetarum tempora rationem ſequuntur ſeſquiplicatam mediarum a Sole diſtantiarum. Si enim tum ſemiaxis major terreſtris orbitae, tum periodicum telluris tempus unitatis loco accipiantur, prodibit aequalitas $D = T^{\frac{2}{3}}$. Hinc igitut erit

T	. . . . =	$75\frac{1}{2}$ annis
log. T	. . . . =	1, 8779469
log. $T^{\frac{2}{3}}$	. . . . =	1, 2519646

proindeque $T^{\frac{2}{3}} = D = 17,\ 8634$, nimirum media Telluris a Sole diſtantia ſe habet ad diſtantiam Cometae mediam a Sole uti 10000 ad 178634. Quoniam vero diſtantia hujus Cometae perihelia, ſeu minima ſumitur $= 0,\ 583$, hac dempta a mediocri 17, 8634 relinquitur Cometae excentricitas $= 17,\ 2804$. Addita porro excentricitate ſemiaxi Ellipſeos majori oritur diſtantia Cometae maxima ſeu aphelia $= 35,\ 1438$. Inventa jam excentricitate e, & ſemiaxe majore D elicitur ſemiaxis minor δ ex nota aequatione $\delta = \sqrt{(D^2 - e^2)}$ hunc in modum

$log.\ e$	. . . . =	1, 2375538
$log.\ e^2$	. . . . =	2, 4751076
e^2	. . . . =	298, 612
$log.\ D$	. . . . =	1, 2519646
$log.\ D^2$	. . . . =	2, 5039292
D^2	. . . . =	319, 102
$D^2 - e^2$	. . . . =	20, 490
$log.\ (D^2 - e^2)$	. . . . =	1, 3115420
$log.\ \sqrt{(D^2 - e^2)}$	. . . . . =	0, 6557710

Fit itaque ſemiaxis minor = 4, 5266. Hiſce porro quantitatibus exploratis & cognitis pronum erit expeditumque Cometae orbitam graphice delineare.

16. In Actis Lipſienſibus nuper propoſitum fuit demonſtratione ſuppreſſa Theorema elegantiſſimum hiſce verbis conceptum: *Data diſtantia Cometae a Sole SM* $= v$, *una cum angulo SMN* $= \psi$, *quem ea cum tangente facit orbitae parabolicae, erit tempus quo ex M ad perihelium pertingit* $= (\cos.\ \psi - \frac{2}{3} \cos.\ \psi^3)\ v \sqrt{2\ v}$, *quod tempus exprimitur per arcum circuli radio* $= 1$ *deſcripti, qui metitur motum medium Solis pro eodem temporis intervallo*. Quum hujus pulcherrimi Theorematis

demonſtrationem quam fieri poteſt ſimpliciſſimam identidem animo volutarem, eandem demum ſic aſſequutus ſum:

Dem: Poſito angulo *PSM*, ſive anomalia vera $= \varphi$, perſpicuum eſt, ſectoris parabolici *PMS* elementum eſſe $\frac{1}{2} v^2 d\varphi$; quum autem in Parabola angulus *SMN* aequetur angulo *SNM*, fiet iccirco $\varphi = 180° - 2\psi$, & $d\varphi = -2d\psi$; proindeque $\frac{1}{2} v^2 d\varphi = -v^2 d\psi$.

Igitur ſector parabolicus $PMS = -\int v^2 d\psi$. Jamvero conſtat ex Conicis, parabolae parametrum p aequalem eſſe quadruplo radio vectori ducto in quadratum ſinus anguli *SMN*, videlicet $p = 4v\,fin.\psi^2$, & conſequenter $v^2 = \frac{p^2}{16\,fin.\psi^4}$: ergo $PMS = -\frac{p^2}{16}\int\frac{d\psi}{fin.\psi^4}$.

Eſt autem, ut ex Calculo Integrali notum eſt, $\int\frac{d\psi}{fin.\psi^4}$

$= -\frac{cos.\psi}{3\,fin.\psi^3} - \frac{2\,cos.\psi}{3\,fin.\psi}$: quapropter PMS

$= \frac{p^2}{16}\left(\frac{cos.\psi}{3\,fin.\psi^3} + \frac{2\,cos.\psi}{3\,fin.\psi}\right) = v^2\left(\frac{1}{3}\,cos.\psi\,fin.\psi + \frac{2}{3}\,cos.\psi\,fin.\psi^3\right)$, ſine ulla Conſtantis additione quia

evaneſcit Sector ubi $\psi = 90°$ Conſtat porro ex Mechanicis, tempus, quo Cometes deſcribit arcum *MP*, aequari duplae areae *MPS* diviſae per radicem ſemiparametri: hinc erit quaeſitum tempus $= \frac{2\,v^2}{\sqrt{\tfrac{1}{2}p}} \times$

$$\left(\tfrac{1}{3}\,cos.\psi\,ſin.\psi + \tfrac{2}{3}\,cos.\psi\,ſin.\psi^3\right) = \frac{2\,v^2}{\sqrt{2\,v\,ſin.\psi^2}} \times$$

$$\left(\tfrac{1}{3}cos.\psi\,ſin.\psi + \tfrac{2}{3}cos.\psi\,ſin.\psi^3\right) = \left(\tfrac{1}{3}cos.\psi + \tfrac{2}{3}cos.\psi\,ſin.\psi^2\right) \times$$

$v\sqrt{2v} = (cos.\psi - \tfrac{2}{3}\,cos.\psi^3)\,v\sqrt{2v}$. *Q. E. D.* Demonſtrationem parum ab hac diſcrepantem inveni poſtmodum in eximio Euleriano Opuſculo, *Recherches & Calculs ſur la vraie Orbite Elliptique de la Comete de l'an.* 1769. *& ſon tems périodique, executées ſous la direction de Mr. Leonhard* EULER *par les ſoins de Mr.* LEXELL. A' St. Petersbourg. 1770.

17. Si in hujus expreſſione temporis ſubſtituatur loco $cos.\psi^3$ ejus valor $\tfrac{3}{4}\,cos.\psi + \tfrac{1}{4}\,cos.\,3\,\psi$, fit tempus ipſum, quo Cometes ad perihelium pertingit, $= (cos.\psi - \tfrac{1}{3}\,cos.\,3\,\psi)\,v\sqrt{\tfrac{1}{2}v}$. Caeterum ſi ſectoris parabolici *PMS* quadraturam aliter inveſtigemus, nimirum ex proprietate parabolae de quadrato ordinatae rectangulum ex abſciſſa in parametrum aequante; aliam tunc nanciſcimur ſectoris ipſius expreſſionem, videlicet

$\frac{1}{p}\left(\frac{1}{6}v^3 \mathit{ſin}. 2\psi^3 + \frac{1}{8}p^2 v \mathit{ſin}. 2\psi\right)$, quae loco ipſius p poſito ejus valore $4v\,\mathit{ſin}.\,\psi^2$ abit in

$$\left(\frac{1}{3}v^2\,\mathit{ſin}.\,\psi\,cos.\,\psi^3 + v^2\,\mathit{ſin}.\,\psi^3\,cos.\,\psi\right) =$$
$$v^2\left(\mathit{ſin}.\,\psi\,cos.\,\psi - \frac{2}{3}\mathit{ſin}.\,\psi\,cos.\,\psi^3\right),$$

hacque ducta in $\frac{2\sqrt{2}}{\sqrt{p}}$, ſive in $\frac{\sqrt{2}}{\sqrt{v\,\mathit{ſin}.\,\psi^2}}$, oritur tempus quaeſitum $= (cos.\,\psi - \frac{2}{3}cos.\,\psi^3)\,v\sqrt{2v}$ prorſus ut ante.

18. Inventa expreſſione temporis ſimpliciſſima, & ad calculos prompte ſupputandos aptiſſima, quaeri jam poteſt qualis eſſe debeat angulus ψ, ſub quo Cometes manente diſtantia v eadem longiſſimum tempus inſumit, ut ad perihelium deſcendat. Capiatur nempe expreſſionis ipſius differentiale $- d\psi\,\mathit{ſin}.\psi + 2d\psi\,\mathit{ſin}.\psi\,cos.\,\psi^2$, quod nihilo aequatum praebet $2\,cos.\psi^2 - 1 = 0$, ſeu $cos.\psi = \mathit{ſin}.\psi = \sqrt{\frac{1}{2}}$. Igitur quaeſitus angulus $\psi = 45^\circ$, ſeu ſemirectus. Eapropter tunc Cometes ad perihelium tardiſſime pertingit, cum ejus a Sole diſtantia cum orbitae tangente angulum ſemirectum conſtituit. Si porro ſubſtituatur $\sqrt{\frac{1}{2}}$ loco $cos.\,\psi$ in expreſſione temporis, prodit tempus longiſſimum $= \frac{2}{3}v\sqrt{v}$, qua ſane formula nihil excogitari poteſt elegantius. Tempus autem *Maximum* ſive longiſſimum hic haberi inde liquet, quod

ſecunda illius differentia negativum praeſefert valorem; ſumpto enim iterum differentiali quantitatis $- d\psi\,\text{ſin.}\,\psi + 2\,d\psi\,\text{ſin.}\,\psi\ \text{cos.}\,\psi^2$ habetur $- d\psi^2\ \text{cos.}\,\psi + 2d\psi^2\ \text{cos.}\,\psi^3 - 4d\psi^2\ \text{ſin.}\,\psi^2\ \text{cos.}\,\psi$, ſeu
$d\psi^2\,(2\ \text{cos.}\,\psi^3 - 4\,\text{cos.}\,\psi\ \text{ſin.}\,\psi^2 - \text{cos.}\,\psi)$,
ubi poſito $\text{cos.}\,\psi = \text{ſin.}\,\psi = \sqrt{\tfrac{1}{2}}$ fit demum
$(\sqrt{\tfrac{1}{2}} - 2\sqrt{\tfrac{1}{2}} - \sqrt{\tfrac{1}{2}})\,d\psi^2$, hoc eſt $- 2d\psi^2\sqrt{\tfrac{1}{2}}$, cujus valor manifeſto negativus maximum ſeu lentiſſimum indicat tempus.



DISQUISITIO VIII.

De Axibus Aequilibrii.

1. Notum eſt, vocari a Mechanicis *Axem Æquilibrii* figurae cujuslibet planae rectam illam, quae per centrum gravitatis figurae ipſius trajicitur, & *Planum Æquilibrii* figurae ſolidae cujuſcumque nuncupari illud, quod per gravitatis centrum ipſius ſolidi tranſit: quae ſane denominatio petita eſt ab aequilibrio, quod tuentur & ſervant partes omnes figurarum planarum ſolidarumque circa illos Æquilibrii Axes, aut Plana. Nimirum ea eſt Axis, vel Plani Æquilibrii proprietas, ut figuram dividat in ſegmenta bina, quae aequalibus hinc inde librantur momentis. Atque haec momentorum aequalitas evidentiſſima eſt, ac nulla peculiari indiget demonſtratione, quotieſcumque Axis, aut Planum Æquilibrii eam obtinet poſitionem, ut ſingula figurae elementa biſariam ſecentur, ſed non aeque evidens ac perſpicua eſt quoties iſta conditio in Axe Æquilibrii deſideratur. Ut igitur (quod Mechanici paſſim ſilentio praetereunt) id ipſum oſtendatur in illis etiam caſibus, in quibus Axis, aut Planum Æquilibrii ſingula fi-

gurae elementa non ſecat bifariam, duo ſeligam exempla, alterum trianguli, alterum coni, in quorum primo Axis, in poſtremo Planum Æquilibrii poſitionem habent baſi parallelam, adeoque diverſam a poſitione requiſita; oſtendamque in hac quoque Axis poſitione bina trianguli, & coni ſegmenta aequalibus momentis circa ipſum Axem librari.

Tab. II. Fig. 6. 2. Sit itaque primo triangulum quodvis BAC & per eius gravitatis centrum E agatur recta MN baſi BC parallela; ajo, bina trianguli ſegmenta MAN, $BMNC$ momentis aequalibus circa Æquilibrii Axem MN librari.

Ducantur enim in triangulo MAN rectae FG, fg, & in trapezio $BMNC$ rectae PQ, pq infinite propinquae, & Axi MN parallelae, agaturque in ipſas a trianguli vertice perpendiculum Ai; tum dic MN a, AN b, FG y, & erit $AT = \frac{by}{a}$, $Tt = \frac{bdy}{a}$, & trianguli AMN elementum $FGgf = \frac{bydy}{a}$. Hoc autem ductu in diſtantiam TH ab Axe MN oritur illius momentum reſpectu ipſius Axis, hoc eſt $\frac{bydy}{a}\left(b - \frac{by}{a}\right)$, cujus integrale $\frac{b^2y^2}{2a} - \frac{b^2y^3}{3a^2}$ praebet momentum trian-

guli *FAG* reſpectu Axis *MN*. Si in hac porro expreſſione fiat $y = a$, prodibit $\frac{1}{2} b^2 a - \frac{1}{3} b^2 a = \frac{1}{6} b^2 a$, quod exhibet momentum totius trianguli *MAN* reſpectu Axis *MN*. Id ipſum conſequuti eſſemus ducta trianguli *MAN* area $\frac{1}{2}\, a b$ in diſtantiam ſui centri gravitatis a latere *MN* ſive in $\frac{1}{3}\, b$.

Sit modo in trapezio *BMNC* recta $PQ = z$, & invenietur $AI = \frac{bz}{a}$, $Ii = \frac{b\,dz}{a}$, elementum trapezii $PQqp = \frac{bz\,dz}{a}$, cujus productum in diſtantiam *IH*, ſive in $\frac{bz}{a} - b$ praebet $\frac{b^2 z^2 dz}{a^2} - \frac{b^2 z\,dz}{a} =$ momento ejus elementi reſpectu Axis *MN*. Summa momentorum hujuſmodi, hoc eſt integrale illius expreſſionis invenitur $= \frac{b^2 z^3}{3a^2} - \frac{b^2 z^2}{2a} +$ Conſt. $=$ momento trapezii *MPQN*; & quoniam evaneſcente trapezio, ſeu facta $z = a$, evaneſcit ejus momentum, idcirco oritur Conſt. $= \frac{1}{2} b^2 a - \frac{1}{3} b^2 a = \frac{1}{6} b^2 a$, & ipſius trapezii momentum $= \frac{b^2 z^3}{3a^2} - \frac{b^3 z^2}{2a} + \frac{1}{6} b^2 a$. Abeunte vero *PQ* in *BC* ſive z in $\frac{3}{2} a$ invenitur totius trapezii *MNCB* momentum $= \frac{9}{8} b^2 a - \frac{9}{8} b^2 a + \frac{1}{6} b^2 a = \frac{1}{6} b^2 a$. Igitur trianguli *MAN*, & trapezii *BMNC* momenta reſ-

pectu Axis MN inter ſe aequantur. $Q. E. D.$

3. Sit ſecundo Conus BAC, & per centrum gravitatis E trajiciatur planum MN baſi BC parallelum: dico momentum coni MAN reſpectu plani MN aequale eſſe momento fruſti conici $BMNC$ reſpectu plani ejuſdem. Ductis planis hinc FG, fg, inde PQ, pq infinite proximis, & baſi parallelis, & Ai in eadem plana perpendiculari, fiat circuli MN diameter $= a$, circuli FG diameter $= y$, perpendiculum $AH = b$, ſitque $1 : \pi$ ratio diametri ad circuli peripheriam. Erit jam $AT = \frac{by}{a}$, $Tt = \frac{b\,dy}{a}$, circulus $FG = \frac{1}{4}\pi y^2$, coni MAN elementum $FGgf = \frac{\pi b y^2 dy}{4a}$, quod ductum in TH, ſive in $b - \frac{by}{a}$ dat ipſius momentum reſpectu Plani MN, nimirum $\frac{\pi b^2 y^2 dy}{4a} - \frac{\pi b^2 y^3 dy}{4a^2}$, & omnium hujuſmodi momentorum ſumma, hoc eſt

$$S\frac{\pi b^2 y^2 dy}{4a} - S\frac{\pi b^2 y^3 dy}{4a^2} = \frac{\pi b^2 y^3}{12a} - \frac{\pi b^2 y^4}{16a^2}$$

praebet momentum coni FAG reſpectu Plani MN. Abeunte autem FG in MN, ſeu y in a oritur $\frac{1}{12}\pi b^2 a^2 - \frac{1}{16}\pi b^2 a^2 = \frac{1}{48}\pi b^2 a^2 =$ momento totius coni MAN reſpectu Plani MN. Idem invenitur ducta coni

MAN ſoliditate $\frac{1}{12}\pi b a^2$ in diſtantiam ſui centri gravitatis a baſi *MN*, hoc eſt in $\frac{1}{4} b$.

Rurſus in cono truncato *BMNC* circuli *PQ* diameter vocetur z, eritque $AI = \frac{bz}{a}$, $Ii = \frac{b\,dz}{a}$, $HI = \frac{bz}{a} - b$, circulus $PQ = \frac{1}{4}\pi z^2$, elementum coni truncati $PQqp = \frac{\pi b z^2 dz}{4a}$, hujus momentum reſpectu Plani $MN = \frac{\pi b z^2 dz}{4a}\left(\frac{bz}{a} - b\right)$. Accepta ſumma omnium momentorum hujuſmodi $\int \frac{\pi b z^2 dz}{4a}\left(\frac{bz}{a} - b\right)$, habetur $\frac{\pi b^2 z^4}{16 a^2} - \frac{\pi b^2 z^3}{12 a} + \text{Conſt.} =$ momento coni truncati *PMNQ*: quum autem una cum cono truncato *PMNQ* evaneſcat ejus momentum, abeunte nimirum z in a, fit ideo $\text{Conſt.} = \frac{1}{12}\pi b^2 a^2 - \frac{1}{16}\pi b^2 a^2 = \frac{1}{48}\pi b^2 a^2$; proinde fruſti ipſius conici momentum evadit $\frac{\pi b^2 z^4}{16 a^2} - \frac{\pi b^2 z^3}{12 a} + \frac{1}{48}\pi b^2 a^2$. Facta poſito $z = \frac{4}{3} a$, ſeu abeunte *PQ* in *BC* (eſt enim, ex proprietate centri gravitatis, baſeos *BC* diameter $= \frac{4}{3} a$), oritur totius fruſti conici *BMNC* momentum $= \frac{16}{81}\pi b^2 a^2 - \frac{16}{81}\pi b^2 a^2 + \frac{1}{48}\pi b^2 a^2 = \frac{1}{48}\pi b^2 a^2$.

Igitur conus MAN, & fruſtum conicum $BMNC$ momentis aequalibus circa Planum MN librantur. Q. E. D.

4. Hinc luculenter patet quam abſurda & praepoſtera ſit centri gravitatis definitio, quae in Mechanicis Inſtitutionibus a nonnemine proponitur, eſſe nimirum in figura qualibet gravitatis centrum punctum illud, per quod traductum utcumque planum figuram dividit in partes duas aequaliter ponderantes, & conſequenter in partes binas aequales, ſi figura ex materia conſtet homogenea; cui porro definitioni multorum Theorematum alias comprobatorum inaedificantur demonſtrationes, prorſus mendoſae & fallaces. Id palam fit in triangulo BAC, ubi Axis aequilibrii MN triangulum ſecat in partes binas MAN, $BMNC$ aequalibus hinc inde libratas momentis, ſed minime aequalibus ponderibus: eſt enim triangulum MAN ad triangulum BAC uti quadratum MN ad quadratum BC, ſive ex proprietate centri gravitatis uti 4 ad 9; ac proinde dividendo, triangulum MAN ſe habet ad trapezium $BMNC$ uti 4 ad 5. Igitur triangulum & trapezium inaequalia ſunt, & inaequaliter ponderantia.

5. Idem in cono quoque perſpicuum fit: nam ob conorum MAN, BAC ſimilitudinem, primus ſe habet ad ſecundum uti cubus diametri baſeos MN, ad cubum diametri baſeos BC, ſive (ex nota proprietate

centri gravitatis in cono) uti cubus numeri ternarii ad cubum quaternarii, hoc eſt uti 27 ad 64; itaque dividendo, erit conus *MAN* ad fruſtum conicum *BMNC* uti eſt 27 ad 37. Ex quo patet, conum ipſum *MAN*, & fruſtum conicum *BMNC* aequalibus quidem hinc inde urgeri momentis, & circa planum *MN* aequilibrari, ſed inaequalibus maſſis & ponderibus donari.

6. Caeterum a non vulgaribus cultioris Mechanicae Scriptoribus de centro gravitatis, quod in quolibet punctorum gravium vel maſſarum ſyſtemate invenitur, tria haec demonſtrari conſuevere; 1.° quod ſi per centrum illud agatur planum aliquod, maſſarum ad unam plani partem exiſtentium pondera in ſuas ab illo diſtantias ducta tantum efficiant quantum pondera maſſarum ad aliam plani partem collocatarum, ducta ſimiliter in ſuas ab ipſo plano diſtantias: 2.° quod ſi maſſae omnes totius ſyſtematis fuerint ad eandem alicujus dati plani partem, producta ex earum maſſarum ponderibus in ſuas ab eo diſtantias aequalia ſint ei quod oritur multiplicando ſummam ponderum per diſtantiam communis centri gravitatis a plano propoſito: 3.° quod ſi maſſae nonnullae fuerint ad unam cujuslibet dati plani partem, reliquae ad partem alteram, ductis ponderibus illarum in ſuas a plano diſtantias, itidemque ponderibus iſtarum in diſtantias ſuas ab eodem plano,

differentia productorum aequalis ſit facto ex aggregato ponderum omnium ducto in diſtantiam communis centri gravitatis a plano eodem.

7. Ex hiſce porro ſtatim conſequitur, in figura quavis geometrica *Planum Æquilibrii* ea donari proprietate, ut bina figurae ſegmenta ipſum interjacentia & quomodocumque inaequalia momentis aequalibus circa illud librentur; at quoniam quaeri merito poteſt, & ſaepe quaeſitum eſt a nonnullis, ut id ipſum de *Plano aequilibrii*, ſive utcumque per centrum gravitatis traducto oſtendatur per ſe, & non ut corollarium ex praedictis centri gravitatis proprietatibus inferatur; propterea quaeſitae demonſtrationis ſpecimen in triangulo, & cono exhibuimus; ex quo deinceps ad idem in propoſita quacumque figura oſtendendum patefacta erit via.



DISQUISITIO IX.

De Curvis a Centro Gravitatis Descriptis.

DUM quaelibet geometrica magnitudo motu aliquo fertur vel rotatorio, vel utcumque curvilineo, vel etiam rectilineo; centrum gravitatis ipſius certas quaſdam deſcribit ſemitas, quarum peculiares a Staticae Scriptoribus recenſentur proprietates. Harum princeps ac celeberrima ea eſt, quae ab inventore GULDINO Guldiniana appellatur, licet eam ſubobſcure PAPPUS Alexandrinus, *Collectionum Mathematicarum* Libro VII. judicaverit: in motu ſcilicet magnitudinis cujuſcumque talis eſt via a centro gravitatis deſcripta (quomodo haec perpendicularis ubique fuerit ad magnitudinem motam), ut ſi viam ipſam ducas in magnitudinem generantem habeas factum aequale figurae motu illo genitae ac productae. Sed praeter ſemitas haſce Mechanicis dudum notas & excultas, dantur Curvae aliae quaedam, quas centrum gravitatis deſcribit dum a geometrica figura partes quaedam demuntur aliae poſt alias donec figura evaneſcat. Curvarum hujuſmo-

di aequationes quatuor elegantiſſimas aſſequutus, eas hic totidem Problematibus expono, demonſtrationeſque Geometris facile occurrentes praetermitto. Eſto itaque

PROBLEMA I.

Tab.II. Fig. 7. Si a dato Circulo *SEG* demantur per vices & ex eadem parte ſectores minimi *BCD*, *DCE*, &c. centrum gravitatis areae deinceps reſiduae iter conficiet curvilineum *CIFO*, cujus initium eſt circuli centrum, finis punctum *O* in ſemidiametro minimi ſectoris ultimo ſuperſtitis, cujus puncti diſtantia a circuli centro trientes duos ſemidiametri exaequat. Quaeritur Curvae *CIFO* natura.

SOLUTIO.

Referatur Curva ad centrum Circuli *C* tamquam ad focum, dicaturque *y* ordinata, ſeu radius vector *CF*, *x* Circuli arcus *BG* ab hac ordinata *CF*, & poſtrema *CO* productis interceptus; & dati circuli radius accipiatur unitati aequalis: Ajo, naturam Curvae per hanc aequationem elegantiſſimam repraeſentari

$$y = \frac{2\ ſin.\ x}{3\ x}.$$

PROBLEMA II.

Si a dati Circuli peripheria *FMI* auferantur ordinatim, & ex eadem parte arcus minimi *FG*, *GI*, &c. centrum gravitatis peripheriae deinceps reſiduae viam percurret curvilineam *CARF*, incipientem a centro circuli, & in extremum ſemidiametri deſinentem. Petitur hujus ſemitae indoles. Tab.II. Fig. 8.

SOLUTIO.

Accipe, ut antea, Circuli centrum *C* pro ſemitae curvilineae foco, & dic ordinatam *CR* $= y$, abſciſſam ſeu circularem arcum *FE* $= x$; circuli radium 1. Æquationem curvae ſimpliciſſimam hanc habebis.

$$y = \frac{\text{ſin.}\ x}{x}.$$

PROBLEMA III.

Tab.II. Fig. 9. Si a dato Circulo QAP per chordas LN, LO, LP &c. abſcindantur ſingillatim, & ex eadem ſemper parte ſegmenta minima LNM, LON, LPO &c. ita ut gravitatis centrum ſegmentorum circuli deinceps ſuperſtitum recedendo a centro C per Curvam incedat $CUFL$ quouſque ad ſemidiametri extremum L pertingat; quaeritur in hac hypotheſi ejuſdem Curvae $CUFL$ aequatio.

SOLUTIO.

Servatis praecedentibus denominationibus radii vectoris $CF = y$, arcus circularis $LA = x$, & ſemidiametri $= 1$, invenitur quaeſita Curvae ad focum C relatae aequatio, quae ſequitur.

$$y = \frac{2\, ſin.^3\, x}{3x - 3\, ſin.x\, cos.x}.$$

PROBLEMA IV.

Si Sphaera QPA genita ex converſione circuli Tab.III. QPA circa diametrum planis ſecetur LN, LO, LP, Fig.10. &c., abſcindentibus per vices & ex eadem parte ſegmenta minima LNM, LON, LPO, &c. quaeritur via $CUFL$, quam tenet centrum gravitatis ſegmentorum ſphaerae deinceps reſiduorum dum a ſphaerae centro C ad ſuperficiem uſque in L progreditur.

SOLUTIO.

Poſito rurſus radio vectore $CF = y$, circuli maximi arcu $LA = x$, ſphaerae ſemidiametro $= 1$, Curvae aequatio umbilicalis ita reperitur

$$y = \frac{6\cos.^2 x - 3\cos.^4 x - 3}{12\cos. x - 4\cos.^3 x - 8}.$$

SCHOLION GENERALE.

1.° Æquatio primi Problematis $y = \frac{2 ſin. x}{3x}$ evaneſcente arcu x mutatur in $\frac{0}{0}$: attamen tunc y eſt $= \frac{2}{3}$, ut conſtat. Sed ſumpto arcu infiniteſimo dx invenitur revera $y = \frac{2 ſin. dx}{3 dx} = \frac{2 dx}{3 dx} = \frac{2}{3}$, quum perſpectum alias ſit ſinum arcus infiniteſimi dx non differre ab ipſo arcu dx.

2.° Pariter ſecundi Problematis aequatio $y = \frac{ſin. x}{x}$, quae transformatur in $\frac{0}{0}$ poſito $x = 0$, per eandem ſubſtitutionem evadit $y = \frac{ſin. dx}{dx} = \frac{dx}{dx} = 1$, qualem eſſe oportet.

3.° Ad aequationem quod attinet Problematis tertii, ſumpto ibi, uti prius, arcu infiniteſimo dx, reperitur $y = \frac{2 ſin.^3 dx}{3 dx - 3 ſin. dx}$ (exiſtente ſcilicet $cos. dx = 1$); vel $y = \frac{2 dx^3}{3 dx - 3 dx} = \frac{2 dx^2}{0}$, quae ſane expreſſio ſuſpecta & deceptrix neutiquam praebet $y = 1$, quem-

admodum opus eſſet. Scrupulus evellitur, ſi conſideres, aequationis numeratorem quantitatem eſſe infniteſimam ordinis tertii, & in fractionis denominatore arcum minimum x differre a ſinu ſuo quantitate pariter infiniteſima tertii ordinis: quare neceſſe erit ad functionum circularium ſeries confugere, ut nodus ſolvatur. Notum eſt, haberi

$$ſin. dx = dx - \frac{dx^3}{2.3} + \frac{dx^5}{2.3.4.5.} - \&c.$$

$$cos. dx = 1 - \frac{dx^2}{2} + \frac{dx^4}{2.3.4.} - \&c.$$

Neglectis igitur differentialibus ultra ordinem tertium, fiet $ſin.^3 dx = dx^3$, & $ſin. dx\ cos. dx = dx - \frac{2}{3} dx^3$. Quapropter aequatio $y = \frac{2\ ſin.^3\ dx}{3\ dx - 3\ ſin. dx\ cos. dx}$ convertitur in $y = \frac{2\ dx^3}{3\ dx - 3\ dx + 2\ dx^3} = \frac{2dx^3}{2dx^3} = 1$, uti oportet.

4.° Quarta denique aequatio $y = \frac{6\ cos.^2 x - 3\ cos.^4 x - 3}{12\ cos. x - 4\ cos.^3 x - 8}$, quae, ſive pro x aſſumatur dx, ſive nihilum, abit in $y = \frac{0}{0}$, generali legi pariter ſubjicitur hoc pacto: Conſtat, eſſe

$$cos. dx = 1 - \tfrac{1}{2} dx^2 + \tfrac{1}{24} dx^4 - \&c.$$

Itaque contemptis differentialibus ultra quartum gradum fiet

$$6\cos.^2 dx = 6 - 6\,dx^2 + 2\,dx^4$$
$$-3\cos.^4 dx = -3 + 6\,dx^2 - 5\,dx^4$$
$$-3 = -3.$$

Quare aequationis numerator evadit $-3dx^4$. Rurſus

$$12\cos. dx = 12 - 6dx^2 + \tfrac{1}{2}dx^4$$
$$-4\cos.^3 dx = -4 + 6dx^2 - \tfrac{7}{2}dx^4$$
$$-8 = -8.$$

Ergo etiam aequationis denominator reperitur $-3dx^4$. Proinde evaneſcente arcu x oritur $y = \frac{-3\,dx^4}{-3\,dx^4} = 1$, uti res poſtulat.



DISQUISITIO X.

DE SINGULARIBUS QUIBUSDAM CENTRI GRAVITATIS AFFECTIONIBUS IN SPATIO HYPERBOLICO-ASYMPTOTICO.

1. CEntri gravitatis inveſtigatio in Spatio Hyperbo- Tab.III. lico-asymptotico quaedam exhibet mira prorſus ac ſin- Fig.11. gularia, quae dum ingenium acuunt & exercent imaginandi vim ſimul magnopere oblectant. Ea breviter attingam, plura huc ſpectantia vel affinia alio tempore daturus.

2. Inveniendum ſit primo centrum gravitatis Spatii Hyperbolico-asymptotici indeterminati *BAMN* a binis ordinatis *BA*, *MN* intercepti. Ducta ordinata *nm* alteri *NM* infinite proxima, conſtat ex Statica, diſtantiam centri gravitatis a recta *BA* inveniri ſi quodlibet ſpatii elementum *NMmn* multiplicetur per diſtantiam ſuam ab ipſa recta *BA*, & omnium hujuſmodi productorum ſumma dividatur per ſummam elementorum.

Eſto itaque

$$EB = b,$$
$$BA = a,$$
$$BN = x,$$
$$NM = y,$$
$$\text{angulus } E = \varphi.$$

Hyperbolae proprietas aequationem praebet $ab = y\,(b + x)$; elementum vero $NMmn$ invenitur $= y\,dx\,ſin.\,\varphi = \frac{ab\,dx\,ſin.\,\varphi}{b + x}$, hocque ductum in diſtantiam ND a recta BA, ſive in $x\,ſin.\,\varphi$ prodit $= \frac{abx\,dx\,ſin.^2\,\varphi}{b + x}$.

Summa omnium iſtiuſmodi productorum, ſeu $\int \frac{abx\,dx\,ſin.^2\,\varphi}{b + x}$ reperitur $= abx\,ſin.^2\,\varphi - ab^2\,ſin.^2\,\varphi \times log.\,(b + x) +$ Conſt.; quae quidem ſumma quum evaneſcat una cum x, oritur Conſt. $= ab^2\,ſin.^2\,\varphi\,log.\,b$, ac proinde $\int \frac{abx\,dx\,ſin.^2\,\varphi}{b + x} = abx\,ſin.^2\,\varphi - ab^2\,ſin.^2\,\varphi \times log.\,\frac{b + x}{b}$. Eſt porro elementorum ſumma $\int y\,dx \times ſin.\,\varphi = \int \frac{ab\,dx\,ſin.\,\varphi}{b + x} = ab\,ſin.\,\varphi\,log.\,(b + x)$

$+ \text{Conſt.} = ab\,ſin.\varphi\,log.\frac{b+x}{b}$. Si itaque ſumma prior dividatur per hanc, quotiens $\frac{x\,ſin.\varphi}{log.(b+x)-log.b}$ $-b\,ſin.\varphi$ exprimit diſtantiam quaeſitam centri gravitatis ab ordinata prima BA.

3. Occurrit hic primo quaedam antilogiae ſpecies quando ponitur $x=0$, in qua ſane hypotheſi evaneſcente ſpatio $ABnm$ debet etiam diſtantia ipſius centri gravitatis a recta BA evaneſcere: attamen iſtius diſtantiae expreſſio $\frac{x\,ſin.\varphi}{log.(b+x)-log.b}-b\,ſin.\varphi$ abit tunc in $\frac{0}{0}-b\,ſin.\varphi$, quae porro quantitas longe abeſſe videtur a nihilo.

4. Ad arcendam antilogiam, capio in fractione $\frac{x\,ſin.\varphi}{log.(b+x)-log.b}$ differentiale tum numeratoris, tum denominatoris, & fractio ipſa convertitur in $\frac{dx\,ſin.\varphi}{dx:(b+x)}=(b+x)ſin.\varphi=b\,ſin.\varphi$ ubi fuerit $x=0$: quamobrem in hac hypotheſi expreſſio $\frac{x\,ſin.\varphi}{log.(b+x)-log.b}$ $-b\,ſin.\varphi$ mutatur in $b\,ſin.\varphi-b\,ſin.\varphi$, hoc eſt in

nihilum, quemadmodum rei natura poſtulat.

5. Eliminatur rurſus antilogia, ſi pro x accipiatur non nihilum abſolutum, ſeu o, ſed infiniteſima magnitudo dx: tunc enim expreſſio $\frac{x\, ſin.\, \varphi}{log.\,(b+x) - log.\, b}$ $- b\, ſin.\, \varphi$ evadit $\frac{dx\, ſin.\, \varphi}{log.\,(b+dx) - log.\, b} - b\, ſin.\, \varphi$; cumque ſit $log.\,(b+dx) = log.\, b + \frac{dx}{b} - \frac{dx^2}{2b^2} + \frac{dx^3}{3b^3}$ $- \frac{dx^4}{4b^4} + \&c. = log.\, b + \frac{dx}{b}$, orietur eadem expreſſio $= \frac{dx\, ſin.\, \varphi}{log.\, b + \frac{dx}{b} - log.\, b} - b\, ſin.\, \varphi = b\, ſin.\, \varphi$ $- b\, ſin.\, \varphi = o$, uti oportet.

6. Quaero nunc in ſpatio infinito $ABST$ diſtantiam centri gravitatis a recta BA. Capio igitur x infinitam in formula $\frac{x\, ſin.\, \varphi}{log.(b+x) - log.\, b} - b\, ſin.\, \varphi$, quae iccirco abit in $\frac{x\, ſin.\, \varphi}{log.\, x}$. Porro fractio haec numeratorem habet infinitum, infinitumque pariter denominatorem, ſed hunc in immenſum minorem quam illum; conſtat enim, logarithmum infiniti numeri 1 + 1 + 1 + 1. + &c. in inf. ſeriem praebere numerorum na-

turalium reciprocam $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. in inf., quarum quidem ſerierum poſterior haec harmonica infinita eſt, uti alias liquet, ſed ſimul infinities minor eſt quam ſeries parallela unitatum, ut ſeriei progreſſum attendenti perſpicuum ſit. Quum igitur fractionis $\frac{x\, ſin.\, \varphi}{log.\, x}$ numerator infinitus ſit, & denominator pariter infinitus, ſed hic infinities minor quam ille, prodit iccirco fractionis valor infinitus. Quamobrem centrum gravitatis ſpatii hyperbolico-asymptotici infiniti *BATS* intervallo diſtat infinito ab ejus origine *BA*.

7. Pergo porro, & inveſtigo diſtantiam centri gravitatis Spatii *BAMN* ab asymptoto *ES*. In elemento *NMmn* a puncto ejus medio *F*, hoc eſt a centro gravitatis ipſius duco normalem *FG* in asymptotum *ES*, quae normalis invenitur $= \frac{1}{2}\, y\, ſin.\, \varphi = \frac{ab\, ſin.\, \varphi}{2b + 2x}$: elementum vero ipſum *NMmn* inventum eſt $= \frac{ab\, dx\, ſin.\, \varphi}{b + x}$. Quare factum ipſius elementi in diſtantiam ſui centri gravitatis ab asymptoto *ES*, nimirum in *FG* prodibit $= \frac{a^2 b^2\, dx\, ſin.^2\, \varphi}{2\,(b + x)^2}$. Capio nunc horum factorum ſummam $\int \frac{a^2 b^2\, dx\, ſin.^2\, \varphi}{2\,(b + x)^2}$, & nanciſcor $- \frac{a^2\, b^2\, ſin.^2\, \varphi}{2\, b + 2\, x}$

$+$ Conſt. Determinatur Conſt., poſita $x = 0$, qua in hypotheſi evaneſcit ſumma factorum, & fit Conſt. $=$

$\frac{1}{2} a^2 b\, \text{ſin.}^2 \varphi$; ac proinde $\int \frac{a^2 b^2 dx\, \text{ſin.}^2 \varphi}{2(b+x)^2} = \frac{1}{2} a^2 b\, \text{ſin.}^2 \varphi$

$- \frac{a^2 b^2\, \text{ſin.}^2 \varphi}{2b+2x}$. Diviſa porro hac quantitate per elementorum ſummam

$$\int \frac{abdx\, \text{ſin.} \varphi}{b+x} = ab\, \text{ſin.} \varphi\, \log. (b+x) + \text{Conſt.}$$

$= ab\, \text{ſin.} \varphi\, \log. \frac{b+x}{b}$, oritur quaeſita diſtantia

$$= \frac{a\, \text{ſin.} \varphi}{2 \log.(b+x) - 2 \log.b} - \frac{ab\, \text{ſin.} \varphi}{(2b+2x)[\log.(b+x) - \log.b]}$$

$$= \frac{ax\, \text{ſin.} \varphi}{(2b+2x)[\log.(b+x) - \log.b]}.$$

8. Evaneſcente ſpatio *BAMN*, ſeu facto $x = 0$, centri gravitatis diſtantia $\frac{ax\, \text{ſin.} \varphi}{(2b+2x)[\log.(b+x) - \log.b]}$ convertitur in $\frac{0}{0}$, quum tamen evidens ſit, diſtantiam illam tunc aequari perpendiculo, quod a puncto medio originis *BA* ipſius ſpatii cadit in asymptotum *ES*, quod ſane perpendiculum reperitur $= \frac{1}{2} a\, \text{ſin.} \varphi$.

9. Hanc rurſus ambiguitatem amoveo ope differentiationis tum numeratoris, tum denominatoris, ex qua habetur

$$\frac{a\,dx\,\text{ſin.}\,\varphi}{2dx[\log.(b+x)-\log.b]+\frac{dx}{b+x}(2b+2x)}$$

$$=\frac{a\,\text{ſin.}\,\varphi\,(b+x)}{(2b+2x)[1+\log.(b+x)-\log.b]}.$$ Haec porro quantitas in hypotheſi $x=0$ manifeſto evadit $\frac{1}{2}a\,\text{ſin.}\varphi$, quemadmodum oportet.

10. Idem nanciſcimur ſumpta pro x quantitate infiniteſima dx: hac enim loco x ſubrogata in praecedenti formula deprehenditur $\frac{a\,dx\,\text{ſin.}\,\varphi}{(2b+2dx)[\log.(b+dx)-\log.b]}$

$=\frac{a\,dx\,\text{ſin.}\,\varphi}{2b\,[\log.(b+dx)-\log.b]}$, quae $\left(\text{ob } \log.(b+dx) = \log.b+\frac{dx}{b}-\frac{dx^2}{2b^2}+\&c. = \log.b+\frac{dx}{b}\right)$ convertitur in

$$\frac{a\,dx\,\text{ſin.}\,\varphi}{2b\left(\frac{dx}{b}+\log.b-\log.b\right)}=\tfrac{1}{2}a\,\text{ſin.}\,\varphi.$$

11. Tranſeo nunc ad diſtantiam exquirendam centri gravitatis in Spatio infinito *BAST* ab asymptoto

ES. Aſſumo itaque x infinitam in formula

$$\frac{a x\, ſin.\, \varphi}{(2b + 2x)\,[\log.(b + x) - \log. b]},$$ quae evidenter degenerat in $\frac{a x\, ſin.\, \varphi}{2 x \log. x} = \frac{a\, ſin.\, \varphi}{2 \log. x}$: quum vero $\frac{a\, ſin.\, \varphi}{2 \log. x}$ quantitas ſit manifeſto infniteſima, conſequens eſt, centrum gravitatis Spatii infiniti Hyperbolico-asymptotici *ABST* abeſſe infinito intervallo ab origine *BA*, proindeque ab asymptoto una *EQ*, infiniteſimo ab asymptoto altera *ES*.



DISQUISITIO XI.

De Maximis et Minimis.

1. PRaeſtantiſſima *Maximorum Minimorumque* Doctrina, quae nobiliorem utilioremque Differentialis Calculi partem conſtituit, nuperis hiſce temporibus tot tantiſque ſummorum Geometrarum laboribus exculta eſt, inventis ditata, & amplificata acceſſionibus, ut ad ſummum perfectionis faſtigium perducta jam eſſe videatur. Attamen, tametſi uberrima rerum copia pulcherrimarum, quibus haec abundat Doctrina, nihil ferme ſpei relinquat, huic abſolutiſſimae Theoriae adhuc poſſe nonnihil addi, ſpicilegium aliquod inter tam amplas meſſes ei faciendum ſupereſt, qui acrius ac ſollertius Theoriam hanc in illis quantitatis variabilis functionibus experiri voluerit, quae originem ſortiuntur ex functionibus aliis per ſe invicem multiplicatis, diviſis, & ad quamcumque poteſtatem elevatis. Itaque operae pretium me facturum exiſtimavi, ſi id ipſum in *Functionibus* primo *Rationalibus* utcumque

multiplicatis, diviſis, &c. periclitarer, novis ſubinde detectis conditionibus ac regulis Maxima ac Minima functionum iſtarum definientibus, perſecuturus poſthac argumentum nobile ac ſplendidum in *Functionibus* quoque *Irrationalibus*, ac *Tranſcendentibus*, quae copioſius ſpicilegium pollicentur. Caeterum quando haec Maximorum Minimorumque Methodus ad ea Problemata ſolvenda adhibetur, in quibus quaeruntur Lineae Curvae Maximi alicujus Minimive proprietate gaudentes efficitur illud in nova Analyſi mirificum ac prope ſingulare, quod ab inventore LA GRANGE *Calculi Variationum* nomine inſignitum fuit; in quo ſcilicet quantitas, quae maxima aut minima eſſe debet, ita ad Curvam totam refertur, ut mutato quolibet ejus Curvae elemento, & ipſa mutetur. Et quamvis Problemata hujuſmodi ante inventum Variationum Calculum Solutione haudquaquam carerent, ut videre eſt in abſolutiſſimo Euleriano Opere de *Maximis ac Minimis*; attamen magno ipſo *EULERO non diffitente* non omnia ſolutionem ſat commodam vel ex toto analyticam admittebant, & nonnulla analyſim vulgarem omnino reſpuere, vel certe tranſcendere videbantur, ut hinc etiam appareat quam longe abſit, ut Differentialis Calculus cum Calculo Variationum confundi citra piaculum poſſit.

Eſto X functio quaecumque τȣ̃ x, ſive algebraica, ſive tranſcendens; ſi in ea ponatur ubivis $x + \varphi$ loco τȣ̃ x, functio ipſa X recipit hanc formam

$$X + \frac{\varphi dX}{dx} + \frac{\varphi^2 ddX}{2dx^2} + \frac{\varphi^3 dddX}{2.3dx^3} + \frac{\varphi^4 d^4 X}{2.3.4dx^4} + \&c.$$

DEMONSTRATIO.

Conſtat ex Calculi Differentialis principiis, functionem X poſito $x + dx$ loco τȣ̃ x abire in $X + dX$, & rurſus in $X + dX$ poſito $x + dx$ pro x augeri $X + dX$ differentiali ſuo $d.(X + dX)$, ſeu fieri $X + 2dX + ddX$, atque in hac iterum pro x poſito $x + dx$ prodire $X + 2dX + ddX + d.(X + 2dX + ddX)$, hoc eſt $X + 3dX + 3ddX + dddX$; ſicque porro deinceps. Atque hi quidem τȣ̃ X valores reſpondent ipſius x valoribus $x + dx$, $x + 2dx$, $x + 3dx$ in X ſubſtitutis (*a*). Oritur igitur tabella:

(*a*) Eſto $F.x$ functio quaecumque ipſius x, ponaturque $x + dx$ pro x; ea abit in $F.(x + dx)$. In hac iterum pro x pone $x + dx$, habebiſque $F.(x + dx + dx)$, ſeu $F.(x + 2dx)$; quod quidem ipſum obveniſſet, ſi in priori $F.x$ protinus poſuiſſes $x + 2dx$ loco τȣ̃ x. Sique iterum in $F.(x + 2dx)$ pones $x + dx$ pro x, obveniet $F.(x + 3dx)$; neque aliud prodiiſſet, ſi in $F.x$ poſuiſſes ab initio $x + 3dx$ loco ipſius x. Eadem eſt de reliquis ratiocinatio.

Valores τ8	— Valores reſpondentes τ8
x	X
$x + dx$	$X + dX$
$x + 2dx$	$X + 2dX + ddX$
$x + 3dx$	$X + 3dX + 3ddX + dddX$
$x + 4dx$	$X + 4dX + 6ddX + 4dddX + d^4X$
$x + 5dx$	$X + 5dX + 10ddX + 10dddX + 5d^4X + d^5X$
$x + 6dx$	$X + 6dX + 15ddX + 20dddX + 15d^4X + 6d^5X + d^6X$
$x + 7dx$	$X + 7dX + 21ddX + 35dddX + 35d^4X + 21d^5X + 7d^6X + d^7X$
&c.	&c.

Nemo non videt, coefficientes valorum τ8 X congruere cum coefficientibus terminorum Binomii Newtoniani ad datam poteſtatem elevati. Ac proinde abeunte x in $x + ndx$ functio X abibit in formulam univerſalem

$$X + ndX + \frac{n.n - 1.ddX}{2} + \frac{n.n - 1\ n - 2.dddX}{2.\ 3}$$

$$+ \frac{n.n-1.n-2.n-3.d^4X}{2.\ 3.\ 4} + \&c.$$

Si jam fuerit n numerus infinitus, ſive omni aſſignabili major, evaneſcent in factoribus ſingulis $n-1$, $n-2$, $n-3$, &c. numeri -1, -2, -3, &c. prae infinito n, ac Formula contrahetur in ſequentem

$$X + ndX + \frac{n^2ddX}{2} + \frac{n^3dddX}{2.3} + \frac{n^4d^4X}{2.3.4} + \frac{n^5d^5X}{2.3.4\ 5}$$

$$+ \frac{n^6d^6X}{2.3.4.5.6} + \&c.$$

Ponatur finita quantitas $ndx = \phi$, adeoque $n = \frac{\phi}{dx}$ ac fiat ſubſtitutio. Induet functio X ex poſitione $x+ndx$, ſeu $x+\varphi$ loco ipſius x praeſcriptam formam

$$X + \frac{\varphi dX}{dx} + \frac{\varphi^2ddX}{2dx^2} + \frac{\varphi^3dddX}{2.3dx^3} + \frac{\varphi^4d^4X}{2.3.4dx^4}$$

$$+ \frac{\varphi^5d^5X}{2.3.4.5dx^5} + \&c.$$

Q. E. D.

12. Cor. Si quantitas negativa fuerit, hoc eſt $-\varphi$, praecedentis Formulae termini pares negativo ſigno afficientur, impares poſitivo, quod eſt per ſe manifeſtum. Igitur forma generaliſſima, in quam abit functio

X ex poſitione $x \pm \varphi$ pro x, repraeſentatur per

$$X \pm \frac{\varphi dX}{dx} + \frac{\varphi^2 ddX}{2dx^2} \pm \frac{\varphi^3 dddX}{2.3dx^3} + \frac{\varphi^4 d^4 X}{2.3.4dx^4}$$

$$\pm \frac{\varphi^5 d^5 X}{2.3.4.5dx^5} + \&c.$$

SCHOLION I.

3. Theorematis Formula aliquam habet analogiam cum Bernoulliana, quam videre eſt in *Act. Lipſ. ann.* 1694, & in *Oper. Joh. BERNOULLII tom.* 1. *n.* 21. Statuit BERNOULLIUS N quantitatem quomodocumque compoſitam ex indeterminatis & conſtantibus, invenitque

$$\int N dz = z\left(N - \frac{z dN}{2\,dz} + \frac{z^2 ddN}{2.3dz^2} - \frac{z^3 dddN}{2.3.4dz^3} + \&c.\right).$$

Indirecta utitur methodo ad id oſtendendum BERNOULLIUS, quod directe nulloque negotio ex notiſſimis Integralis Calculi ſcitis licet eruere quando N functio eſt τȣ̃ z. Cum enim ſit $\int N dz = Nz - \int z dN$; ſi fiat $dN = N' dz$, erit rurſus $\int z dN$, ſive $\int N' z dz$

$= \frac{z^2 N'}{2} - \int \frac{z^2\, dN'}{2}$. Poſito iterum $dN' = N'' dz$, oritur $\int \frac{z^2\, dN'}{2}$, ſeu $\int \frac{N'' z^2\, dz}{2} = \frac{z^3 N''}{2.3} - \int \frac{z^3 dN''}{2.3}$: factoque rurſus $dN'' = N''' dz$, habetur $\int \frac{z^3\, dN''}{2.3}$, ſeu $\int \frac{N''' z^3\, dz}{2.3} = \frac{z^4 N'''}{2.3.4} - \int \frac{z^4\, dN'''}{2.3.4}$; ſicque porro. Subrogatis hiſce valoribus in priori formula $\int Ndz$, ea invenietur $= z\left(N - \frac{zN'}{2} + \frac{z^2 N''}{2.3} - \frac{z^3 N'''}{2.3.4} + \&c.\right)$, hoc eſt aſſumpto dz conſtanti, $\int Ndz = z\left(N - \frac{zdN}{2dz} + \frac{z^2 ddN}{2.3dz^2} - \frac{z^3 dddN}{2.3.4dz^3} + \&c.\right)$. Sed majorem habet formula noſtra affinitatem cum Maclauriniana, quam videſis in eximio Opere *Treatiſe of Fluxions* §. 751., ubi expreſſa y per ſeriem $A + Bz + Cz^2 + Dz^3 + \&c.$, in qua conſtantes ſingulae A, B, C, &c. poni etiam poſſunt, ſi opus fuerit, nihilo aequales, ſi evaneſcente z dicatur E valor τȣ y, fluat vero uniformiter z, inve-

nitur $y = E + \frac{z\,dE}{dz} + \frac{z^2 ddE}{2\,dz^2} + \frac{z^3\,dddE}{2.3\,dz^3} + \&c.$ Maclauriniana demonſtratio ita ſe habet: Quoniam $y = A + Bz + Cz^2 + Dz^3 + \&c.$, facto $z = 0$, orietur $y = A$, ſive ex hypotheſi $= E$. Sumptis vero fluxionibus erit $\frac{dy}{dz} = B + 2Cz + 3Dz^2 + \&c.$, & evaneſcente z fiet $B = \frac{dy}{dz} = \frac{dE}{dz}$. Acceptis rurſus fluxionibus prodit $\frac{ddy}{dz^2} = 2C + 2.3\,Dz + \&c.$, poſitoque $z = 0$, habetur $C = \frac{ddy}{2\,dz^2}$, ſive $= \frac{ddE}{2\,dz^2}$. Erutis iterum fluxionibus colligitur $\frac{dddy}{dz^3} = 2.3\,D + \&c.$, & facto $z = 0$, fit $D = \frac{dddy}{2.3\,dz^3}$, ſeu $= \frac{dddE}{2.3\,dz^3}$. Sicque porro, coefficientes reliqui determinantur. Igitur $y = A + Az + Cz^2 + Dz^3 + \&c. =$

$$E + \frac{z\,dE}{dz} + \frac{z^2\,ddE}{2\,dz^2} + \frac{z^3\,dddE}{2.3\,dz^3} + \&c.$$

Demonſtrationem aliam elegantiſſima ſyntheſi adornavit idem Maclaurinus §. 255. Id ipſum jampridem invenerat Taylorus in profundiſſima Diſquiſitione de *Methodo Incrementorum*.

S c h o l i o n II.

4. Si X exprimatur generali formula $A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 +$ &c., quod fieri ſemper poteſt vel accurate, vel quamproxime ut liqnet, alia ſuccurrit brevior Theorematis noſtri demonſtratio. Nam ſubſtituendo $x + \varphi$ pro x, abit formula in $A + B(x+\varphi) + C(x+\varphi)^2 + D(x+\varphi)^3 + E(x+\varphi)^4 + F(x+\varphi)^5 +$ &c. $=$

$$+ A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c. (M)$$
$$+ B\varphi + 2Cx\varphi + 3Dx^2\varphi + 4Ex^3\varphi + 5Fx^4\varphi + \&c. (N)$$
$$+ C\varphi^2 + 3Dx\varphi^2 + 6Ex^2\varphi^2 + 10Fx^3\varphi^2 + \&c. (O)$$
$$+ D\varphi^3 + 4Ex\varphi^3 + 10Fx^2\varphi^3 + \&c. (P)$$
$$+ E\varphi^4 + 5Fx\varphi^4 + \&c. (Q)$$
$$+ F\varphi^5 + \&c. (R).$$

Eſt vero, ut patet,

$$(M) = X,$$

$$(N) = \frac{\varphi d X}{dx},$$

$$(O) = \frac{\varphi^2 dd X}{2 dx^2},$$

$$(P) = \frac{\varphi^3 ddd X}{2.3\, dx^3},$$

$$(Q) = \frac{\varphi^4 d^4 X}{2.3.4\, dx^4},$$

$$(R) = \frac{\varphi^5 d^5 X}{2.3.4.5\, dx^5}, \ \&c.$$

poſita x aequabiliter fluente, hoc eſt $ddx = 0$. Igitur $A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c.$

$$= X + \frac{\varphi d X}{dx} + \frac{\varphi^2 dd X}{2 dx^2} + \frac{\varphi^3 ddd X}{2.3\, dx^3}$$

$$+ \frac{\varphi^4 d^4 X}{2.3.4\, dx^4} + \frac{\varphi^5 d^5 X}{2.3.4.5\, dx^5} + \&c.$$

Si φ negativo ſigno accipiatur, locorum parium termini hoc eodem ſigno affecti prodibunt. Inde vero fiet ge-

neraliter $A + B(x \pm \varphi) + C(x \pm \varphi)^2 + D(x \pm \varphi)^3$

$$+ \&c. = X \pm \frac{\varphi dx}{dx} + \frac{\varphi^2 ddX}{2dx^2} \pm \frac{\varphi^3 dddX}{2.3\, dx^3} + \&c.$$

S c h o l i o n III.

5. Quantitas φ, quam finitam accepimus, aſſumi etiam poteſt utcumque parva, immo finita qualibet quantitate in immenſum minor, hoc eſt infiniteſima. Ad hoc enim ſatis eſt ſumere dx infiniteſimam ſecundi ordinis (quod late conceſſum, ut conſtat); tuncque fiet ndx, ſeu φ infiniteſima ordinis primi, qualem nos in poſterum habebimus.

D e f i n i t i o I.

6. Dicitur X *Functio Uniformis* τῦ x, ſi valoribus ſingulis determinatis ipſius x ſinguli reſpondent determinati valores, non geminati, non plures Functionis X.

7. Cor. ~~Functiones~~ omnes Algebraicae Rationales, ſive Integrae, ſive Fractae, Uniformes ſunt. Integrarum formula generalis eſt $a + bx + cx^2 + ex^3 + fx^4 + gx^5 + \&c.$, Fractarum vero eſt

$$\frac{a + bx + cx^2 + ex^3 + fx^4 + \&c.}{a' + b'x + c'x^2 + e'x^3 + f'x^4 + \&c.}$$; atque ejuſmodi expreſſiones, quemcumque variabili x valorem

tribuas, nonniſi unicum valorem praebent, reſpondentem valori unico ipſius x.

DEFINITIO II.

8. Vocatur X *Functio Multiformis* τȣ̃ x, ſi pro unico valore determinato in locum x ſubſtituto, non unicum, ſed plures determinatos valores praeſefert.

9. Cor. 1. Functiones omnes Irrationales Multiformes ſunt ob multiplicem valorem terminorum radicalium, quibus quantitas variabilis involvitur.

10. Cor. 2. Functio X Biformis, Triformis, Quadriformis, &c. erit, ſi ea determinetur per aequationem ſecundi, tertii, quarti, &c. gradus; ac generatim n:formis erit eadem X, ſeu pro unico valore τȣ̃ x valores diverſos praeſeferet numero n, ſi ea definiatur per aequationem $X^n - MX^{n-1} + NX^{n-2} - PX^{n-3} + QX^{n-4} - \&c. = 0$; in qua quidem aequatione quantitates M, N, P, Q, &c. functiones deſignant uniformes τȣ̃ x, eſtque M, ut compertum alias, ſumma omnium valorum Functionis X, N ſumma factorum ex binis, P ſumma factorum ex ternis, Q ſumma factorum ex quaternis, &c.

11. Cor. 3. Functiones quoque Tranſcendentes vel Uniformes eſſe poſſunt, vel Multiformes. Quin etiam

in hiſce habentur & *Infinitiformes*: nam Functiones Tranſcendentes *A ſin.x*, *A cos.x*, *A tang.x*, *A cot.x*, *A ſec.x*, *A coſec.x*, in quibus *A* deſignat arcum circuli habentem *x* ſinum, vel coſinum, &c., Infinitiformes ſunt ob infinitos arcus uni eidemque valori ipſius *x* reſpondentes.

12. Cor. 4. Si Functio Multiformis *X* unicum tantum exhibeat valorem realem pro valore quovis unico ipſius *x*, reliquis exiſtentibus imaginariis, ea inſtar functionis Uniformis habebitur. Ita generatim $\left(a + bx + cx^2 + ex^3 + fx^4 + gx^5 + \&c.\right)^{\frac{m}{n}}$ functio eſt ſpecie Multiformis, re Uniformis, ſi *n* ſit numerus impar, *m* vero ſive par, ſive impar; in hoc enim caſu unicus eſt, ut conſtat, Functionis valor realis, reliqui omnes imaginarii. Quod ſi *n* numerus fuerit par, Functio eadem vel Biformis erit, vel Nulliformis ob valorem ejus realem vel geminum, vel nullum in hac hypotheſi, dummodo fractio $\frac{m}{n}$ minimis terminis fuerit expreſſa.

DEFINITIO III.

13. Functio *X Maximum* ſortitur valorem *A* ex ſubſtitutione valoris determinati *a* in locum variabilis

x, ſi A major fuerit tum valoribus A', A'', A''', &c. proxime antecedentibus, tum valoribus $'A$, $''A$, $'''A$, &c. proxime ſequentibus, quos ſucceſſive acquirit eadem X ex aliquantulo tum incremento, tum decremento variabilis x ultra a. Viciſſim Functio X *Minimum* obtinet valorem A reſpondentem valori determinato a in locum τȣ x ſubrogato, ſi A minor fuerit quam valores A', A'', A''' &c. proxime antecedentes, & $'A$, $''A$, $'''A$, &c. proxime conſequentes, in quos ſucceſſive abit ipſa X ſi variabilis x vel aliquantulum augeatur ultra a, vel aliquantulum minuatur. Eſto exgr. $X = \frac{a^3}{x^2 - 2bx + b^2 + c^2}$, quae oritur *Maxima* ponendo $x = b$, fitque $X = \frac{a^3}{c^2}$; ſi enim variabilem x tantillum ſive augeas, ſive minuas ultra b, ſumaſque $x = b \pm \omega$ (accepta ω utcumque parva) fiet ſemper $X = \frac{a^3}{c^2 + \omega^2} < \frac{a^3}{c^2}$. Si viciſſim fuerit $X = x^2 - 2fx + f^2 \pm g^2$, ea *Minima* evadet, nimirum $= \pm g^2$, poſito f pro x; propterea quod accepta x aliquantillum ſive majori, ſive minori quam f, videlicet facta $x = f \pm \omega$, oritur in utroque caſu $X = \pm g^2 + \omega^2 > \pm g^2$.

PRÆNOTATIO.

14. Deſignantibus P, Q, R, T, &c. functiones rationales integras τῦ x, & p, q, r, t, &c. numeros integros poſitivos, atque etiam nihilum, Functio noſtra X (quam hic rationalem integram ſupponimus) ad ſex formas ſequentes opportune ac commode revocabitur:

1.° $X = P$;

2.° $X = PQ$;

3.° $X = PQRT$ .... &c.;

4.° $X = P^p$;

5.° $X = P^p Q^q$;

6.° $X = P^p Q^q R^r T^t$ .... &c.

PARS I.

De Maximis ac Minimis Functionum Rationalium Integrarum.

PROBLEMA I.

15. Invenire Maximum, ac Minimum valorem Functionis integrae $X = P$.

SOLUTIO.

Ad hoc praeſtandum invenire oportet valorem determinatum variabilis x, qui loco τῦ x in Functione X, vel P ſubrogatus eam reddat Maximam, Minimamve. Sit igitur 1:o x talis qualis oportet, ut valor Functionis X *Maximus* fiat. Si jam in hac hypotheſi augeatur x quantitate utcumque parva φ, ac ſubſtituatur $x + \varphi$ loco τῦ x in Functione X, ſeries inde orta

$$(1.)\ X + \frac{\varphi dX}{dx} + \frac{\varphi^2 ddX}{2\,dX^2} + \frac{\varphi^3 dddX}{2.3\,dx^3} + \frac{\varphi^4 d^4 X}{2.3.4\,dx^4}$$

$+$ &c. minor erit (13.), quam X: ac ſimiliter ſi x minuatur utcumque minima quantitate φ, & pro x ponatur ubivis $x - \varphi$ in expreſſione Functionis X, valor ſeriei, in quam abit ipſa X, hoc eſt (2.) $X - \frac{\varphi dX}{dx}$

$$+ \frac{\varphi^2 ddX}{2\,dx^2} - \frac{\varphi^3 dddX}{2.3\,dx^3} + \frac{\varphi^4 d^4 X}{2.3.4\,dx^4} - \text{\&c., minor}$$

iterum erit (13.), quam X. Igitur tam $\frac{\varphi dX}{dx} + \frac{\varphi^2 ddX}{2\,dx^2}$

$$+ \frac{\varphi^3 dddX}{2.3\,dx^3} + \frac{\varphi^4 d^4 X}{2.3.4\,dx^4} + \text{\&c., quam} - \frac{\varphi dX}{dx}$$

$$+ \frac{\varphi^2 ddX}{2\,dx^2} - \frac{\varphi^3 dddX}{2.3\,dX^3} + \frac{\varphi^4 d^4 X}{2.3.4\,dx^4} - \text{\&c. nega-}$$

tiva quantitas erit. Id vero neutiquam eveniret, niſi foret $\frac{\varphi d x}{d x} = 0$; propterea quod poſita φ in immenſum minore quantitate qualibet data, & conſequenter evaneſcentibus prae ipſa φ poteſtatibus altioribus φ^2, φ^3, &c., ſeries $\frac{\varphi d X}{d x} + \frac{\varphi^2 dd X}{2 d x^2} + \frac{\varphi^3 ddd X}{2.3 d x^3} + \frac{\varphi^4 d^4 X}{2.3.4 d x^4}$ + &c. contrahitur in unicum terminum $\frac{\varphi d X}{d x}$, ſimiliterque ſeries $- \frac{\varphi d X}{d x} + \frac{\varphi^2 dd X}{2 d x^2} - \frac{\varphi^3 ddd X}{2.3 d x^3}$ + &c. redigitur ad terminum unicum $- \frac{\varphi d X}{d x}$; quorum terminorum $+ \frac{\varphi d X}{d x}$, $- \frac{\varphi d X}{d x}$ alter ſemper eſt affirmativus, alter ſemper negativus. Ergo in caſu *Maximi* proveniet ſemper $\frac{\varphi d X}{d x}$, ſeu $\frac{d X}{d x} = 0$.

Sit 2°. x talis, qualem poſtulat valor *Minimus* Functionis X, ita ut ſi augeatur, ac minuatur ipſa x quantitate infiniteſima φ, valor uterque Functionis X, ex poſitione $x + \varphi$, & $x - \varphi$ pro x, reſultans, hoc eſt

(1.) $X + \frac{\varphi\, d X}{dx} + \frac{\varphi^2\, dd X}{2\, dx^2} + \frac{\varphi^3\, ddd\, X}{2.3\, dx^3} +$ &c., nec

non $X - \frac{\varphi\, d X}{dx} + \frac{\varphi^2\, dd X}{2 dx^2} - \frac{\varphi^3\, ddd\, X}{2.3. dx^3} + \frac{\varphi^4\, d^4\, X}{2.3.4 dx^4}$

— &c. major ſit (13.), quam X. Ergo tam ſeries

$\frac{\varphi\, d X}{dx} + \frac{\varphi^2\, dd X}{2\, dx^2} + \frac{\varphi^3\, ddd\, X}{2.3\, dx^3} +$ &c., quam ſeries

altera $- \frac{\varphi\, d X}{dx} + \frac{\varphi^2\, ddx}{2\, dx^2} - \frac{\varphi^3\, ddd\, X}{2.3\, dx^3} +$ &c. valorem exhibebit poſitivum; quod tamen fieri nequit, niſi ponatur $\frac{\varphi\, d X}{dx} = 0$, quum ſeries prior aequetur ſoli $\frac{\varphi\, d X}{dx}$, poſterior ſoli $- \frac{\varphi\, d X}{dx}$, quorum quidem valorum alter eſt neceſſario affirmativus, alter neceſſario negativus. Igitur etiam in caſu *Minimi* fiet $\frac{d X}{dx} = 0$.

Facta itaque $\frac{d X}{dx} = 0$, aequationis hujus radices, ſeu valores τῦ x inde eruti, & in expreſſione X ſubrogati, *Maximum*, *Minimumve* (ſi quis erit) Functioni X valorem inducent.

Ut autem nunc diſpiciatur quandonam Functionis X valor *Maximus* erit, quandonam *Minimus*, quandonam neuter (nam & hoc fieri poteſt), attendenda ſedulo eſt ſeries (B) $\pm \frac{\varphi dX}{dx} + \frac{\varphi^2 ddX}{2dx^2} \pm \frac{\varphi^3 dddX}{2.3\,dx^3} + \frac{\varphi^4 d^4X}{2.3.4\,dx^4} \pm \frac{\varphi^5 d^5 X}{2.3.4.5\,dx^5} +$ &c. reſpondens duplici poſitioni $x \pm \varphi$ pro x. Si ſecundus ſeriei terminus $\frac{\varphi^2 ddX}{2dx^2}$, ſubſtituendo valorem τȣ x erutum ex aequatione $\frac{dX}{dx} = 0$, prodeat negativus, adeoque & negativum valorem exhibeat fluxio ſecunda $\frac{ddX}{dx^2}$ (poſita ſemper dx conſtante), Functionis X valor *Maximus* erit, ob valorem nimirum negativum totius ſeriei (B), quae evaneſcente primo termino, & reliquis poſt ſecundum in immenſum prae ipſo decreſcentibus ad ſolum ſecundum terminum redigitur.

Contra Functionis X valor *Minimus* erit, ſi ſecundus ſeriei terminus $\frac{\varphi^2 ddX}{2dx^2}$, adeoque & ſecunda flu-

xio $\frac{ddX}{dx^2}$ poſitivum valorem praebeat ex ſubſtitutione radicis aequationis $\frac{dX}{dx} = 0$; hinc enim fiet, ut ſeries tota (B), quae uni ſecundo termino aequivalet, valorem obtineat poſitivum.

At vero Functionis X valor neque *Maximus* erit, neque *Minimus*, ſi ſecundus ipſe terminus $\frac{\varphi^2 ddX}{2\,dx^2}$, ſeu $\frac{ddX}{dx^2}$ ſubſtituta radice aequationis $\frac{dX}{dx} = 0$ evadat nihilo aequalis, dummodo per eandem ſubſtitutionem una cum ſecundo non evaneſcat quoque tertius ſeriei terminus $\pm \frac{\varphi^3 dddX}{2.3\,dx^3}$: Quippe nulleſcente ex hypotheſi ſecundo termino, ac ſubſiſtente tertio tota ſeries ob terminos reliquos poſt tertium in infinitum prae ipſo decreſcentes aequipollet tantummodo eidem tertio, qui cum ex duplici poſitione $x + \varphi$, & $x - \varphi$ loco τȢ x duplicem praeſeferat valorem $\pm \frac{\varphi^3 dddX}{2.3\,dx^3}$, alterum ſcilicet neceſſario poſitivum, alterum neceſſario negativum, ſeriei ipſi (B) binos oppoſitos valores

inducit. Hoc autem repugnat cum *Maximi*, tum *Minimi* indoli, quorum illud exigit, ut valor uterque ſeriei (B) ex poſitione $x + \varphi$, & $x - \varphi$ pro x oriundus ſit negativus; iſtud vero poſtulat, ut uterque ipſe valor ſit poſitivus.

Quod ſi contingat, ut per ſubſtitutionem radicis aequationis $\frac{dX}{dx} = 0$ una cum ſecundo ſeriei (B) termino tertius quoque nulleſcat, criterium petendum erit ex termino quarto $\frac{\varphi^4 d^4 X}{2.3.4 dx^4}$, vel etiam ex quarta fluxione $\frac{d^4 X}{d x^4}$, quae nimirum, ſi per ſubſtitutionem praedictae radicis valorem praeferat negativum, indicabit *Maximum*; ſi valorem induat poſitivum, indicabit *Minimum*; ſi valorem praebeat nihilo aequalem, dummodo ſubſequens terminus quintus $\pm \frac{\varphi^5 d^5 X}{2.3.4.5 dx^5}$ facta eadem ſubſtitutione non evaneſcat ſimul, neque *Maximum*, neque *Minimum* indicabit.

Ac ſi quintus ipſe ſeriei terminus $\pm \frac{\varphi^5 d^5 X}{2.3.4.5 dx^5}$

per ſubſtitutionem valoris $r_8''' x$ ex aequatione $\frac{dX}{dx} = 0$ eruti evaneſcat una cum quatuor terminis praecedentibus, legem ac normam dabit ſubſequens terminus ſextus $\frac{\varphi^6 d^6 X}{2.3.4.5.6\, dx^6}$, ad quem unice (dummodo & is non evaneſcat) ſeries tota (B) reducitur, terminis reliquis poſt ſextum in infinitum prae ipſo imminutis. Valor quippe negativus ejus termini functionem X de more *Maximam* arguet; poſitivus *Minimam*; nulleſcens, neque *Maximam*, neque *Minimam*, permanente tamen termino ſeptimo $\pm \frac{\varphi^7 d^7 X}{2.3.4.5.6.7\, dx^7}$ alternatim poſitivo, ac negativo.

Nam ſi & terminus ſeptimus una cum praecedentibus per ſubſtitutionem valoris $r_8''' x$ prodeat nihilo aequalis, ex octavo termino $\frac{\varphi^8 d^8 X}{2.3.4.5.6.7.8\, dx^8}$ petenda erit norma; qui ſane terminus ſi per illam ſubſtitutionem oriatur negativus, Functio X habebitur de more *Maxima*; ſi poſitivus, *Minima*: ſi prodeat nullus ſubſiſtatque terminus nonus alterna vice poſitivus, &

negativus $\pm \frac{\phi^{9} d^{9} X}{2.3.4.5.6.7.8.9\, dx^{9}}$, Functio X neque *Maxima* erit, neque *Minima*.

Par erit in ſequentibus ratiocinatio, cujus unicum fundamentum eſt, quod ſeries (B) negativa eſſe debet pro valore *Maximo* Functionis X; poſitiva pro *Minimo*; alternis poſitiva, & negativa pro neutro.

Igitur Functio X *Maxima* erit, vel *Minima*, ſi ſeriei (B) termini ſucceſſivi, ſimul evaneſcentes, fuerint numero impares, ſive unus, ſive tres, ſive quinque &c; *Maxima* quidem, ubi ſeriei (B) terminus poſt ultimum evaneſcentem ſuperſtes prodeat negativus, *Minima* ſi poſitivus. At nec *Maxima* erit, nec *Minima*, ubi ſucceſſivi termini ſimul evaneſcentes ſeriei ejuſdem ſint numero pares, nimirum vel bini, vel quatuor, vel ſex, &c.

Allata eſt ergo methodus inveniendi valores *Maximos*, *Minimoſque* Functionis $X = P$; traditum criterium diſtinguendi inter utroſque, hoſque ab illis internoſcendi; propoſita demum norma ipſorum etiam abſentiam dignoſcendi. *Q. E. I.*

Scholion I.

16. Hinc luculentiſſime patet, aequationem $\frac{dX}{dx} = 0$ ſemper neceſſario locum habere quoties Functio

X Maximum aliquem, *Minimumve* valorem adipiſcitur; at ſimul etiam patet, haudquaquam veram eſſe propoſitionem converſam, *Maximum* ſcilicet, vel *Minimum* aliquem valorem Functioni *X* ſemper neceſſario ineſſe quando aequatio $\frac{dX}{dx} = 0$ locum habet. Nam ſi una cum fluxione hac prima Functionis *X* evaneſcat, per ſubſtitutionem valoris τȣ̃ *x* inde hauſti, fluxio quoque ſecunda; aut cum hiſce duabus evaneſcat tertia ſimul, & quarta; aut cum iſtis quatuor quinta etiam, & ſexta; ſicque numerus par quicumque fluxionum ſucceſſivarum annihiletur, Functio *X* nec Maximum ullum, nec Minimum valorem exhibebit, quantumvis aequationis $\frac{dX}{dx} = 0$ radices in ipſa *X* ſatagas ſubrogare. Itaque dupliciter offendit vulgata methodus plerorumque Geometrarum, magni oppido nominis, ac de Matheſi alias praeclare meritorum, qui aequationem unicam $\frac{dX}{dx} = 0$ normam ſtatuunt ac regulam detegendi Maxima, ac Minima. Ea quippe aequatio 1.° nec ſemper, nec neceſſario Maximum, Minimumve indicat; nec 2.° Maximis, Minimiſque a ſe invicem diſtinguendis par eſt. Ita ſi proponatur ex. gr. functio $X = x^4$

$-(3a+b)x^3+(3a^2+3ab)x^2-(a^3+3a^2b)x$ $+ba^3$, cujus inveſtigandi ſint valores *Maximi*, *Minimive*; facta $\frac{dX}{dx}=0$, ſeu $4x^3-(9a+3b)x^2+(6a^2+6ab)x-a^3-3a^2b=0$, ſeu $x^3-\frac{(9a+3b)x^2}{4}+\frac{(6a^2+6ab)x}{4}-\frac{a^3+3a^2b}{4}=0$, ex hujus aequationis reſolutione prodeunt ipſius radices a, a, $\frac{1}{4}a+\frac{3}{4}b$. Neque tamen propterea radix a in Functione X ſubſtituta Maximum, Minimumve valorem eidem inducit: ex hac quippe ſubſtitutione oritur $X=0$; valor autem hunc proxime antecedens, ponendo $a-\omega$ pro x, eſt $-(a-b-\omega)\omega^3$, ac valor eum proxime ſubſequens, poſito $a+\omega$ loco τȣ x, eſt $(a-b+\omega)\omega^3$, quorum prior, imminuto in immenſum ω, eſt idem cum $-(a-b)\omega^3$, alter cum $(a-b)\omega^3$, ſive alter neceſſario poſitivus, alter negativus, nimirum hic minor valore intermedio $X=0$, ille major; ac proinde valor ipſe intermedius $X=0$ nec Maximus eſſe poteſt, nec Minimus (13); quod a vulgari methodo incertum relinquitur. Id vero ipſum luculenter indicat ſuperior methodus, quum ſecunda fluxio $\frac{ddX}{dx^2}=12x^2-(18a+6b)x+6a^2+6ab$

per ſubſtitutionem τϋ a loco x evaneſcat, tertia autem fluxio $\frac{dddX}{dx^3} = 24x - 18a - 6b$ per eandem ſubſtitutionem haudquaquam evaneſcat abiens in $6a - 6b$, quae aſſumptis, ut ſupponimus, a, & b inaequalibus nequit eſſe nihilo aequalis. Si praeterea in Functione X loco τϋ x radix tertia $\frac{a+3b}{4}$ aequationis $\frac{dX}{dx} = 0$ ſubrogetur, abit functio ipſa in $-\frac{1}{3}\left(\frac{3b-3a}{4}\right)^4$, qui ſane valor Maximus, an Minimus, an neuter ſit, per vulgarem methodum non ſit manifeſtum; adhibito autem criterio ſuperiori ſtatim liquet, valorem hunc Minimum eſſe ob ſecundam fluxionem $\frac{ddX}{dx^2} = \frac{36}{16} \times (a-b)^2$ poſitivam. Vel ex hoc unico exemplo perſpicue intelligitur, quantum vulgari praeſtet methodus in antecedenti Problemate expoſita ac demonſtrata; quae & ſimpliciſſima eſt, & certiſſimis innixa principiis, & evidentiſſimis notionibus deducta, & ad quantitates functioneſque omnium generum latiſſime patens.

Caeterum quum functio quaelibet quantitatis variabilis x per Curvae alicujus ordinatam rite exprima-

tur, ut Geometrae compertum habent, hinc eſt, ut univerſalis Maximorum ac Minimorum methodus cum Problemate inveniendi Maximas, Minimaſque Curvarum ordinatas undequaque conſentiat. Manifeſtum porro eſt ex Curvarum indole ac geneſi, ordinatae Maximae reſpondere radium oſculi, ſeu Evolutae poſitivum, Minimae negativum: Cumque, aſſumpta ordinata X, ac fluente aequabiliter abſciſſa x, radii oſculatoris formula ſit $\dfrac{\left(1+\dfrac{dX^2}{dx^2}\right)^{\frac{3}{2}}}{\dfrac{-ddX}{dx^2}}$, quae exiſtente X Maxima, aut Minima, adeoque $\dfrac{dX}{dx}=0$, abit in $-\dfrac{dx^2}{ddX}$, erit propterea quantitas $-\dfrac{dx^2}{ddX}$ poſitiva in hypotheſi ordinatae Maximae, ~~negativa in~~ hypotheſi Minimae, & conſequenter ddX negativa pro Maxima ordinata, ſeu pro valore Maximo functionis X, poſitiva pro Minima ordinata, ſeu pro Minimo valore functionis ejuſdem; prorſus ut ſupra.

SCHOLION II.

17. Poſita generatim functione integra X, ſeu $P = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + Hx^7 +$ &c., cujus Maxima ac Minima determinantur per radices aequationis $\frac{dP}{dx} = 0$, ſeu $B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + 6Gx^5 + 7Hx^6 +$ &c. $= 0$, ſi hujus aequationis radices reales, & inaequales ſecundum ordinem magnitudinis diſpoſitae dicantur, a, b, c, e, f, &c., ſitque adeo $a > b$, $b > c$, $c > e$, $e > f$, &c., harum unaquaeque functioni P valorem Maximum, vel Minimum tribuet. Etenim poſito V pro factore aequationis $\frac{dP}{dx} = 0$ radices imaginarias, ſi quae ſunt, involvente, habebitur $\frac{dP}{dx} = (x-a) \times (x-b)(x-c)(x-e)(x-f) \ldots V = 0$; ac proinde

$$\frac{ddP}{dx^2} = (x-b)(x-c)(x-e)(x-f)\ldots V$$
$$+ (x-a)(x-c)(x-e)(x-f)\ldots V$$
$$+ (x-a)(x-b)(x-e)(x-f)\ldots V$$
$$+ (x-a)(x-b)(x-c)(x-f)\ldots V$$
$$+ (x-a)(x-b)(x-c)(x-e)\ldots V$$

$+ V \ldots + (x-a)(x-b)(x-c)(x-e)(x-f)\ldots \times \frac{dV}{dx}$: In hac porro aequalitate ſubſtituendo 1.° a loco τȢ x oritur $\frac{ddP}{dx^2} = (a-b)(a-c)(a-e)(a-f)\ldots M$ (abeunte V in M ex illa ſubſtitutione); qui ſane valor, ob M ſemper poſitivum, & factores $a-b$, $a-c$, &c. pariter poſitivos, nequit non eſſe poſitivus: Atque hinc radix prima, ſeu maxima a valorem Minimum functioni P largietur (15.). Subſtituatur 2.° in eadem $\frac{ddP}{dx^2}$ radix ſecunda b pro x; fiet $\frac{ddP}{dx^2} = (b-a)(b-c) \times (b-e)(b-f)\ldots V$, quae ob factorem unum $b-a$ negativum, reliquos autem poſitivos negativa eſt, indicatque icciрco (15.) functionem P Maximam oriri ex radice ſecunda b. Ponatur 3.° pro x radix tertia c; prodibit $\frac{ddP}{dx^2} = (c-a)(c-b)(c-e)(c-f)\ldots V$, ſeu poſitiva ob factores duos $c-a$, $c-b$, negativos, & reliquos poſitivos. Itaque radix tertia c functionem P Minimam reddit (15.). Subſtituatur 4.° radix quarta e; orietur $\frac{ddP}{dx^2} = (e-a)(e-b)(e-c)(e-f)\ldots V$, hoc

eſt negativa ob priores tres factores negativos, reliquos poſitivos; proindeque radix quarta e functionem P Maximam praebet. Subſtituta 5.° ſimiliter radice quinta f, abit $\frac{ddP}{dx^2}$ in $(f-a)(f-b)(f-c)(f-e)\ldots V$, quae ob factores quatuor priores negativos, reliquos autem poſitivos quantitas eſt poſitiva, arguitque propterea (15.) functionis P valorem, ex quinta radice f oriundum, Minimum fore. Eadem porro ratiocinatione inveniemus, tot haberi valores Maximos, Minimoſque in functione P, quot habentur radices reales & inaequales in aequatione $\frac{dP}{dx} = 0$; ita quidem, ut facta radicum ipſarum diſtributione ſecundum ordinem quantitatis earundem, primumque locum maxima, poſtremum minima occupante, ac caeteris ordinatim loca intermedia tenentibus, radices omnes locorum imparium, prima, tertia, quinta &c. Functionem P Minimam exhibeant, radices autem locorum parium, ſecunda, quarta, ſexta, &c. Maximam praebeant. Id ipſum quoque demonſtrabitur, ubi radicum inaequalium a, b, c, e, f, &c. aliquae, vel etiam omnes negativae fuerint. Si enim ex. gr. e, & f negativae accipiantur, hoc

eſt $-e$, $-f$, ſitque ratione diſtributionis $-e > -f$, ac propterea $f > e$; in ...atis expreſſionibus valorum for... ...ae $\frac{ddP}{dx^2}$ mutabuntur ubique ſigna quantitatum e, & f, ſemperque adhibito examine deprehendetur ipſius $\frac{ddP}{dx^2}$ valor poſitivus ex prima radice, negativus ex ſecunda, poſitivus iterum ex tertia, negativus ex quarta, &c.; prorſus ut ante.

At ſi aequatio $\frac{dP}{dx} = 0$ radices habeat aequales numero n, ita ut factor unus aequationis ipſius ſit $(x-h)^n$ deſignante h radicem aequalem, factor vero alter, ſi ullus fuerit, radices inaequales, nec non imaginarias involvens dicatur T, ſitque propterea $\frac{dP}{dx} = 0 = (x-h)^n.T$; invenietur $\frac{ddP}{dx^2} = n\,(x-h)^{n-1}.T + (x-h)^n.\frac{dT}{dx}$; $\frac{d^3P}{dx^3} = n.\,n-1.\,(x-h)^{n-2}.T + 2n\,(x-h)^{n-1}.\frac{dT}{dx} + (x-h)^n.\frac{ddT}{dx^2}$; $\frac{d^4P}{dx^4} = n.\,n-1.\,n-2.\,(x-h)^{n-3}.T$

$$+3n.n-1.(x-h)^{n-2}.\frac{dT}{dx} + 3n(x-h)^{n-1}.\frac{ddT}{dx^2}$$

$$+(x-h)^n.\frac{d^3T}{dx^3}; \frac{d^5P}{dx^5} = n.n-1.n-2.n-3.(x-h)^{n-4}$$

$$+4n.n-1.n-2.(x-h)^{n-3}.\frac{dT}{dx} + 6n.n-1.(x-h)^{n-2}\times$$

$$\frac{ddT}{dx^2} + 4n(x-h)^{n-1}.\frac{d^3T}{dx^3} + (x-h)^n.\frac{d^4T}{dx^4};$$ atque ita porro innoteſcet, fluxionem $(n+1)$:eſimam ipſius P, ſeu $\frac{d^{n+1}P}{dx^{n+1}}$ conſtare ex primo termino $n(n-1)(n-2)\times(n-3)(n-4)(n-5)\ldots(x-h)^{n-n}.T$, ſeu $n(n-1)\times(n-2)(n-3)\ldots\ldots T$, ex ſecundo termino habente factorem $x-h$, ex tertio termino habente factorem $(x-h)^2$, ex quarto, cujus factor erit $(x-h)^3$, denique ex ultimo adaequate $\frac{(x-h)^n d^nT}{dx^n}$. Subrogata nunc in hiſce fluxionis $\frac{d^{n+1}P}{dx^{n+1}}$ terminis radice h pro x, evaneſcunt omnes praeter primum, abeunte $\frac{d^{n+1}P}{dx^{n+1}}$ in $n(n-1)(n-2)(n-3)(n-4)\ldots\ldots T$. Itaque jam

ſit manifeſtum, per radicem aequalem h Maximum aliquem, Minimumve valorem functionis P obtineri (15.), ſi fluxio $\frac{d^{n+1} P}{d x^{n+1}}$ poſt ultimam evaneſcentem ſuperſtes fuerit ordinis paris, hoc eſt ſi numerus n impar fuerit; contra vero functionem P nec Maximam, nec Minimam oriri, ubi fluxio illa gradum imparem habuerit, & conſequenter numerus n fuerit par. Hinc porro palam eſt, quoties aequatio $\frac{dP}{dx} = 0$ radices aequales habeat numero impares, tres, quinque, ſeptem, &c., functionem P ex ſubſtitutione radicis ejuſdem aequalis fieri vel Maximam, vel Minimam; ac viciſſim quotieſcumque praedicta aequatio parem numerum complectatur radicum aequalium, duas, quatuor, ſex, &c. functionem P ex illa ſubſtitutione nec Maximam, nec Minimam evadere.

Quod ſi fuerit vel factor $T = 1$, vel idem T complectatur radices tantum imaginarias; ſeu quod idem eſt, ſi aequationis $\frac{dP}{dx} = 0$ radices omnes vel fuerint reales & aequales, vel reliquae praeter aequales ipſas fuerint imaginariae, praebebit utroque caſu radix aequa-

lis valorem Minimum functionis P, ſuppoſito impari radicum omnium numero; ſuppoſito autem hoc numero pari functio P nec Minima evadet, nec Maxima. Nam exiſtente n numero pari (quo in caſu ex demonſtratis functio P nec Maxima fieri poteſt, nec Minima), par quoque eſſe debet numerus radicum omnium aequationis $\frac{dP}{dx} = 0$, quum reliquarum radicum imaginariarum in factore T contentarum nonniſi par numerus eſſe poſſit, uti ex Algebra conſtat. Itaque nullum erit Maximum, neque Minimum functionis P, ſi par fuerit numerus radicum omnium aequationis $\frac{dP}{dx} = 0$ continentis radices tantum aequales, & reliquas imaginarias. Contra vero, ubi radicum aequalium numerus impar fuerit, atque adeo impar numerus radicum omnium additis ſcilicet reliquis imaginariis factoris T ſemper numero paribus, Minima oritur functio P ex ſubſtitutione radicis aequalis (15.) ob valorem affirmativum fluxionis paris $\frac{d^{n+1}P}{dx^{n+1}} = n(n-1)(n-2) \times (n-3)(n-4)(n-5) \ldots\ldots T$, in qua factor T ex ſubſtitutione quantitatis cujuſvis pro x nunquam non invenitur affirmativus.

Eadem omnino demonſtratione conficitur, ſuppoſito $\frac{dP}{dx} = o = (x-g)^m (x-h)^n (x-l)^r \ldots T$, hoc eſt datis in aequatione $\frac{dP}{dx} = o$ radicibus aequalibus g numero m, aequalibus radicibus h numero n, aequalibus radicibus l numero r, &c., caeteriſque inaequalibus, & imaginariis in factore T contentis, radices ipſas g, h, l, &c. valorem Maximum, Minimumve functioni P conciliare, ſi earum numeri m, n, r, &c. impares fuerint; neutrum dare, ſi pares. Huc uſque dicta in radices aequales poſitivas perinde ac negativas adamuſſim quadrare, notum eſt quam quod maxime. Neque minus perſpicuum eſt, facta ſimul radicum omnium aequalium & inaequalium diſtributione ſecundum ordinem quantitatis radices omnes locorum imparium (ſive aequales eae fuerint, ſive inaequales) functionem P Minimam exhibere, locorum parium Maximam, ubi tamen radices aequales numero impares fuerint, paribus ſcilicet neutrum praebentibus.

18. Ex hactenus demonſtratis ſequentes Canones eximiae utilitatis, maximique uſus deducuntur.

CANON I.

Ad Maxima, aut Minima functionis X detegenda fluxionem ejus primam $\frac{dX}{dx}$ nihilo aequalem ponito, hujusque radices loco x in functione illa ſubrogato: tum facta eadem ſubſtitutione in fluxione ſecunda $\frac{ddX}{dx^2}$ accidet, ut vel 1.° ejus valor prodeat negativus; vel 2.° oriatur affirmativus; vel 3.° ſit nullus, nec talis ſit tamen fluxionis tertiae $\frac{ddd\,X}{dx^3}$ valor ex illa ſubſtitutione proveniens. Si primum contingat, functionem X Maximam credito, ſi alterum, Minimam; ſi tertium, nec Maximam, nec Minimam judicato.

CANON II.

19. At ſi tertiae quoque fluxionis $\frac{ddd\,X}{dx^3}$ valor ex eadem ſubſtitutione evaneſcat, ex quarta fluxione $\frac{d^4\,X}{dx^4}$ criterium petito; ex hujus nimirum valore negativo functionem X Maximam reputato, ex affirmativo Mi-

nimam, ex nullo (ſummodo non evaneſcat ſimul fluxio quinta $\frac{d^5 X}{dx^5}$) nec Maximam, nec Minimam habeto.

Canon III.

20 In univerſum, ſi fluxionum ſucceſſivarum functionis X numerus impar quicumque evaneſcat, functionem ipſam X vel Maximam, vel Minimam pronuncito; Maximam, ſi fluxionis ſuperſtitis poſt ultimam evaneſcentem valor fuerit negativus; Minimam, ſi affirmativus.

Canon IV.

21. At nec Maximam, nec Minimam functionem X habeto, ſi fluxionum ipſius ſucceſſivarum numerus par quicumque nihilo aequalis oriatur.

Canon V.

22. Si aequationis $\frac{dX}{dx} = 0$ radices omnes reales ſecundum ordinem magnitudinis diſponantur, primuſque locus maximae, poſtremus minimae tribuatur, locaque intermedia a radicibus intermediis ordinatim occupentur, functionem X oriri Maximam pro certo habeto ex radicibus inaequalibus locorum parium, Minimam ex

radicibus inaequalibus locorum imparium; idemque judicato de radicibus aequalibus ejuſdem ſpeciei ubi tamen ipſarum numerus impar fuerit.

PROBLEMA II.

Invenire Maxima, aut Minima functionis integrae $X = PQ$.

Solutio.

Patet ex praecedenti Problemate, Maxima aut Minima, ſi quae ſunt, functionis X, ſeu PQ per radices aequationis $\frac{dX}{dx} = 0$, ſeu $\frac{PdQ + QdP}{dx} = 0$ indubitanter obtineri. Maxima autem fiet functio PQ, ſi fluxio ſecunda $\frac{ddX}{dx^2}$, ſeu $\frac{PddQ + 2dPdQ + QddP}{dx^2}$ negativa fuerit; Minima, ſi affirmativa; nec Maxima vero, nec Minima, ſi fluxio eadem nulleſcat, permaneatque fluxio tertia $\frac{d^3X}{dx^3}$, hoc eſt

$$\frac{Pd^3Q + 3dPd^2Q + 3d^2PdQ + Qd^3P}{dx^2}$$, ſicque

porro per Probl. praec. pronum erit inferre, functio-

nem PQ Maximam fore, ubi evaneſcente numero quovis impari fluxionum ſucceſſivarum functionis ipſius fluxio ſubſequens negativa deprehendatur; Minimam; ubi haec prodeat affirmativa; at neque Maximam, neque Minimam, ubi numerus fluxionum earundem evaneſcentium fuerit par. *Q.E.I.*

S c h o l i o n.

24. Calculum ineunti conſtabit, fluxionem quartam ipſius PQ fore $Pd^4Q + 4dPd^3Q + 6d^2Pd^2Q + 4d^3PdQ + Qd^4P$, fluxionem quintam prodire $Pd^5Q + 5dPd^4Q + 10d^2Pd^3Q + 10d^3Pd^2Q + 5d^4PdQ + Qd^5P$. Hinc ad fluxionem n:eſimam functionis PQ obtinendam ſat erit ſeries binas ordinare

$$P,\ dP,\ d^2P,\ d^3P,\ d^4P,\ d^5P, \ldots\ldots, d^nP$$

$$d^nQ, d^{n-1}Q, d^{n-2}Q, d^{n-3}Q, d^{n-4}Q, d^{n-5}Q, \ldots\ldots, Q,$$ ac unius terminos per terminos alterius infra ſibi reſpondentes multiplicare, productiſque ſingulis praefigere coefficientes binomii Newtoniani ad poteſtatem n elevati. Ita fluxio n:eſima functionis PQ fiet $Pd^nQ + ndPd^{n-1}Q$

$$+ \frac{n.n-1}{2} . d^2Pd^{n-2}Q + \frac{n.n-1.n-2}{2.3} . d^3Pd^{n-3}Q$$

$$+ \frac{n.n-1.n.-2.n-3}{2.3.4}\ d^4Pd^{n-4}Q$$

$+ \ldots\ldots\ldots\ldots\ldots + Qd^{n}P$. Eadem ratiocinatione licebit demonſtrare, fluxionem n:eſimam functionis PQR eſſe $Pd^{n}.\, QR + ndPd^{n-1}.\, QR$

$$+ \frac{n.\,n-1}{2}.\, d^{2}Pd^{n-2}.\, QR + \frac{n.\,n-1.\,n-2.}{2.\,3} \times$$

$$d^{3}Pd^{n-3}.\, QR + \frac{n.\,n-1.\,n-2.\,n-3}{2.\,3.\,4}.\, d^{4}Pd^{n-4}.\, QR$$

$+ \ldots\ldots\; QRd^{n}P$, ubi $d^{n}.\, QR$, $d^{n-1}.\, QR$, $d^{n-2}.\, QR$, &c. indicant fluxiones n:eſimam, (n—1):eſimam, (n—2):eſimam functionis QR prius inventas. Haud diſſimili methodo innoteſcet fluxionem n:eſimam functionis $PQRT\ldots$ ex quolibet factorum numero coaleſcentis prodire $Pd^{n}.\, QRT\ldots + ndPd^{n-1}.\, QRT\ldots + \frac{n.\,n-1.}{2} \times$

$$d^{2}Pd^{n-2}.\, QRT\ldots + \frac{n.\,n-1.\,n-2}{2.\,3}.\, d^{3}Pd^{n-3}.\, QRT\ldots$$

$$+ \frac{n.\,n-1.\,n-2.\,n-3}{2.\,3.\,4}.\, d^{4}P\,d^{n-4}.\, QRT\ldots$$

$\ldots\ldots\ldots\ldots + QRT\ldots d^{n}P$.

PROBLEMA III.

25. Functionis $X = PQRT\ldots$ Maxima, aut Minima determinare.

Solutio.

Juxta ſuperius demonſtrata, radices aequationis $\frac{dX}{dx} = 0$, hoc eſt

$$\frac{PQR\ldots dT + PQT\ldots dR + PRT\ldots dQ + QRT\ldots dP}{dx} = 0$$

Maxima, aut Minima functionis $PQRT\ldots$ ſuppeditabunt; Maximum quidem, ſi ſecunda fluxio ipſius $PQRT\ldots$ negativa prodierit; Minimum, ſi affirmativa; neutrum, ſi nulla, ſubſiſtente fluxione tertia ipſius $PQRT\ldots$ At ſi una cum fluxione ſecunda haec quoque tertia evaneſcat, ad criterium jam ante traditum ſemper erit confugiendum, eaque regula conſtantiſſime obſervanda. Fluxiones autem cujuſcumque gradus ipſius functionis $PQRT\ldots$ per regulam in Schol. Problematis praecedentis inventam expedite & commode eruentur. Patet ergo propoſitum. *Q. E. I.*

PROBLEMA IV.

26. Functionis $X = P^p$ valores Maximos Minimosve inveſtigare.

SOLUTIO

Maximi, Minimive functionis P^p valores per radices aequationis $\frac{dX}{dx} = 0$, ſeu $\frac{pP^{p-1}dP}{dx} = 0$ de more determinantur. Hinc diviſa aequatione per pP^{p-1}, radices hujus $\frac{dP}{dx} = 0$ Maximos, Minimoſve, ſi modo adſint, functionis P^p valores praebebunt. Ut autem Maxima a Minimis internoſcantur, conſiderare oportebit fluxionem ſecundam $\frac{ddX}{dx^2}$, ſeu $\frac{p.p-1.P^{p-2}dP^2}{dx^2}$ $+\frac{pP^{p-1}ddP}{dx^2}$, in qua evaneſcente primo termino ob $dP = 0$ ſolus ſupereſt ſecundus; qui icciroo ſi ſubſtituta radice prodit negativus, Maximam indicat functionem P^p; ſi oritur affirmativus, indicat Minimam: atque ubi fuerit p numerus impar, ſeu pP^{p-1} poteſtas gradus paris, adeoque ſemper affirmativa, ex ſolo $\frac{ddP}{dx^2}$ Maximi, Minimique diſtinctio colligi poterit. At vero nec Maximus erit, nec Minimus functionis valor, ſi

evaneſcat terminus $\frac{pP^{p-1}\, ddP}{dx^2}$, permaneatque fluxio tertia $\frac{ddd\,X}{d\,x^3}$, nimirum

$$\frac{p.p-1.p-2.P^{p-3}dP^3+3p.p-1.P^{p-2}dPddP+pP^{p-1}d^3P}{dx^3},$$

hoc eſt $\frac{pP^{p-1}d^3P}{dx^3}$, evaneſcente ſcilicet dP. Hacque porro methodo ut prius (15.) pergendum erit. *Q.E.I.*

S C H O L I O N.

27. Ex aequatione $\frac{pP^{p-1}\,dP}{dx} = 0$ licet inferre, functionis P^p valorem Minimum per radices aequationis $P = 0$ ſemper obtineri, ſi par fuerit numerus p; contra vero nec Maximum, nec Minimum ſi fuerit impar. Eſt quippe fluxio p:eſima ipſius P^p, ut ſupputanti innoteſcet, compoſita ex termino $\frac{p.p-1.p-2.p-3.p-4\ldots 1dP^p}{dx^p}$, nec non ex aliis ordinatim ductis in P^{p-1}, P^{p-2}, P^{p-3}, P^{p-4}, ... P, qui propterea evaneſcunt om-

nes praeter primum. Hinc (15.) functio P^p Maxima evadet vel Minima, ſi par fuerit numerus p, ſeu par fluxio ſuperſtes p: eſima functionis P^p: & quoniam termini illius primi $\frac{p.p-1.p-2.p-3......1\,dP^p}{dx^p}$, vel etiam $\frac{dP^p}{dx^p}$ valor non poteſt non eſſe affirmativus ob potentiam parem fluxionis dP ſemper affirmativam; fiet iccirco P^p Minimum; at nec Maximum nec Minimum exiſtente p impari.

PROBLEMA V.

28. Invenire Maxima, ac Minima functionis P^pQ^q.

SOLUTIO.

Nihilo aequata de more fluxione $\frac{dX}{dx}$ =

$\frac{(qPdQ+pQdP)\;P^{p-1}Q^{q-1}}{dx}$, erit quoque

$\frac{qPdQ+pQdP}{dx}=0$, cujus adeo radices Maximum, Minimumve functioni P^pQ^q valorem inducent; Ma-

ximum quidem, ſi fluxio ſecunda $\frac{ddX}{dx^2}$, quae evaneſcente $\frac{qPdQ + pQdP}{dx}$ reducitur ad $P^{p-1}\ Q^{q-1} \times \left(\frac{qP\,ddQ + pQ\,ddP + (p+q)\,dP\,dQ}{dx^2}\right)$, fuerit negativa; Minimum, ſi affirmativa; neutrum, ſi nulla, permanente fluxione tertia $\frac{dddX}{dx^3} = d^3.\ P^p\ Q^q =$ &c. Ubi autem p, q ſint numeri impares, & conſequenter P^{p-1}, Q^{q-1} poteſtates pares, valoriſque iccirco affirmativi, ex ſola $\frac{qPddQ + pQddP + (p+q)\,dPdQ}{dx^2}$ licebit Maximum a Minimo internoſcere. *Q. E. I.*

Scholion.

29. Ex formula $\frac{(qP\,dQ + pQ\,dP)\ P^{p-1}\ Q^{q-1}}{dx} = 0$ colligitur tum $P = 0$, tum $Q = 0$. Jam vero poſita $P = 0$, fluxio p:^{eſima} functionis $P^p\ Q^q$ inito calculo invenitur $\frac{p.p-1.p-2.p-3.p-4\ldots 1\ Q^q\,dP^p}{dx^p}$, terminis nimirum reliquis ob factores P^p, P^{p-1},

P^{p-2}, P^{p-3}, P^{p-4}, ... P, quos comprehendunt, evaneſcentibus. Itaque (15.) aequationis $P = 0$ radices in functione $P^p Q^q$ ſubrogatae Maximum, Minimumve ſuppeditabunt functionis valorem, ſi numerus p par fuerit; neutrum, ſi impar: & quidem in ipſa hypotheſi p paris habebitur Maximum, ſi Q^q negativa ſit; Minimum, ſi affirmativa, quod ſemper continget ubi par fuerit poteſtas Q^q, ſeu numerus q. Eodem modo aſſumpta $Q = 0$, functionis $P^p Q^q$ fluxio ordinis q:eſimi, quae ſola ſupereſt poſt omnes fluxiones praecedentes nihilo aequales, eſt

$$\frac{q.q-1.q-2.q-3.q-4 \ldots\ldots P^p dQ^q}{dx^q},$$ nulleſcentibus ſcilicet fluxionis iſtius terminis caeteris per Q^q, Q^{q-1}, Q^{q-2}, Q^{q-3}, ... Q multiplicatis. Igitur (15.) per radices aequationis $Q = 0$ Maxima, vel Minima functionis $P^p Q^q$ ſemper invenientur ubi fuerit q numerus par; ubi autem ſit impar, nullum habebitur in functione illa Maximum Minimumve: utroque porro q, & p poſito pari, ob valorem fluxionis praecedentis affirmativum Minima orietur functio $P^p Q^q$.

PROBLEMA VI.

30. Functionis $P^p Q^q R^r T^t$... Maximos, ac Minimos valores inveſtigare.

Solutio.

Facta $\frac{dX}{dx} = 0$, hoc eſt $P^{p-1} Q^{q-1} R^{r-1} T^{t-1} \ldots \times$

$$\left(\frac{tPQR\ldots dT + rPQT\ldots dR + qPRT\ldots dQ + pQRT\ldots dP}{dx}\right)$$

$= 0$, hacque diviſa per $P^{p-1}\ Q^{q-1}\ R^{r-1}\ T^{t-1} \ldots$ obtinentur de more Maxima, ac Minima functionis propoſitae per radices aequationis

$$\frac{tPQR\ldots dT + rPQT\ldots dR + qPRT\ldots dQ + pQRT\ldots dP}{dx}$$

$= 0$: quae ſi dicatur M, ob $\frac{dX}{dx} = P^{p-1} Q^{q-1} R^{r-1} T^{t-1} \ldots M$, erit fluxio ſecunda $\frac{ddX}{dx^2}$ (neglectis terminis per M multiplicatis) aequalis quantitati $\frac{P^{p-1}\ Q^{q-1}\ R^{r-1}\ T^{t-1} \ldots dM}{dx}$, quae facta radicis in locum τοῦ x ſubſtitutione vel negativa prodibit, vel ~~poſitiva, vel nulla~~: Si primum, functio $P^p\ Q^q\ R^r\ T^t \ldots$ (15.) Maxima orietur; ſi alterum, Minima; ſi tertium (dummodo non evaneſcat ſimul fluxio ſubſequens tertia), nec Maxima erit, nec Minima. Quod ſi tertia quoque fluxio evaneſcat, judicium petendum erit ex quarta, uti ante adnotavimus, &c. *Q. E. I.*

SCHOLION.

31. Ex praecedenti aequatione $\frac{dX}{dx} = 0$, obtinetur

$$1.^{\circ}\ P = 0;$$

$$2.^{\circ}\ Q = 0;$$

$$3.^{\circ}\ R = 0;$$

$$4.^{\circ}\ T = 0;\ \&c.$$

Jam vero in hypotheſi $P = 0$, ſubductis calculis paſſim fit, fluxionem p:^{eſimam} functionis $P^p Q^q R^r T^t \ldots$ ſubſtituta radice aequationis $P = 0$ pro x ſuperſtitem fore poſt praecedentes omnes nihilo aequales, eamque ipſam neglectis terminis in P ductis eſſe

$$\frac{p \cdot p - 1 \cdot p - 2 \cdot p - 3 \cdot p - 4 \ldots\ldots 1\ Q^q R^r T^t \ldots dP^p}{dx^p};$$

ex quo infertur (15.), per ſubſtitutionem radicum aequationis $P = 0$ Maximam, aut Minimam evadere functionem $P^p Q^q R^r T^t \ldots$, ſi fuerit p numerus par; neutram, ſi impar; ac Minimam ſpeciatim, ſi etiam q, r, t, &c. pares fuerint. Rurſus aſſumpta aequatione $Q = 0$, hujuſque radicibus loco τȣ͂ x ſubſtitutis in fluxionibus functionis praedictae, evaneſcere deprehenduntur fluxiones illae uſque ad fluxionem q:^{eſimam}, quae

invenitur aequalis ipſi $\frac{q.q-1.q-2.q-3.q-4\ldots 1 P^p R^r T^t\ldots dQ^q}{dx^q}$; quae ſane indicat, exiſtente q impari, functionem $P^p Q^q R^r T^t\ldots$ per ſubſtitutionem radicis erutae ex aequatione $Q = 0$ nec Maximam fieri, nec Minimam; alterutram vero evadere exiſtente q pari, ac ſpeciatim Minimam poſitis quoque reliquis p, r, t, &c. paribus. Ita etiam facta $R = 0$, captiſque fluxionibus memoratae functionis ſucceſſivis, ſubſtitutiſque loco τ8 x radicibus aequationis $R = 0$, prodit fluxio r: eſima, quae fluxionum non evaneſcentium prima eſt,

$$\frac{r.r-1.r-2.r-3.r-4\ldots\ldots\ldots 1 P^p Q^q T^t\ldots dR^r}{dx^r};$$

haec porro functionem $P^p Q^q R^r T^t\ldots$ Maximam aut Minimam arguit poſito r numero pari, ac peculiariter Minimam poſitis etiam p, q, t, &c. paribus; neutram vero facto r impari. Denique eodem modo ſi accipiatur $T = 0$, hujuſque aequationis radices ſubſtituantur in fluxionibus functionis praedictae, fluxio, quae prima ſupereſt praecedentibus aliis evaneſcentibus, eſt ordinis t: eſimi, eaque oritur

$$\frac{t.t-1.t-2.t-3.t-4\ldots\ldots 1 P^p Q^q R^r\ldots dT^t}{dx^t}.$$

Igitur (15.) & in hoc eventu functio $P^p Q^q R^r T^t \ldots$ Maxima aut Minima evadit per radicis illius ſubſtitutionem, ſi t numerus fuerit par; ac ſi alii quoque p, q, r, &c. pares fuerint, functio Minima oritur; at neque Maxima fit neque Minima, ſi t fuerit impar.

PARS II.

De Maximis, ac Minimis Functionum Rationalium Fractarum.

PRÆNOTATIO.

32. SI per P, Q, R, T, &c., nec non F, G, H, L, &c. exprimantur functiones rationales integrae τȢ x, ſintque p, q, r, t, &c., nec non f, g, h, l, &c. numeri integri, vel etiam nihilum; Functio noſtra X (quam hic *rationalem* fractam accipimus) ad ſex haſce formas revocari percommode poterit;

$$1^\circ\ X = \frac{P}{F};$$

$$2^\circ\ X = \frac{P\,Q}{F\,G};$$

$$3^\circ\ X = \frac{P\,Q\,R\,T\ldots}{F\,G\,H\,L\ldots};$$

$$4.^{\circ}\ X = \frac{P^p}{F^f};$$

$$5.^{\circ}\ X = \frac{P^p\,Q^q}{F^f\,G^g};$$

$$6.^{\circ}\ X = \frac{P^p\,Q^q\,R^r\,T^t\ldots}{F^f\,G^g\,H^h\,L^l\ldots}.$$

PROBLEMA I.

33. Maximum, Minimumve Functionis $X = \frac{P}{F}$ valorem determinare.

SOLUTIO.

Eadem, qua ſupra (15.), ratiocinatione fas cuique eſt demonſtrare, Maxima, ac Minima Functionis $\frac{P}{F}$ elici oportere ex aequatione $\frac{dX}{dx}$, ſeu $d.\left(\frac{P}{F\,dx}\right) = 0$, hoc eſt $\frac{F\,dP - P\,dF}{F^2\,dx} = 0$. Hinc ſimplicius orietur $\frac{F\,dP - P\,dF}{dx} = 0$ hujuſque aequationis radices loco τȣ̃ x in Functione $\frac{P}{F}$ ſubrogatae Maximos, Minimoſve (ſi qui fuerint) eidem Functioni valores inducent. Ut autem Maxima a Minimis internoſcantur,

per ratiocinationem ante inſtitutam (15.) radices aequationis $FdP - PdF = 0$ ſubſtituendae ſunt pro x in fluxione ſecunda $\frac{ddX}{dx^2} = dd.\left(\frac{P}{Fdx^2}\right) =$

$d.\left(\frac{FdP - PdF}{F^2 dx^2}\right) = \frac{FddP - PddF}{F^2 dx^2}$

$- \frac{2dF(FdP - PdF)}{F^3 dx^2} =$ (ob evaneſcentem magnitudinem $FdP - PdF$) $\frac{FddP - PddF}{F^2 dx^2}$; ita quidem ut affirmativus valor fractionis $\frac{FddP - PddF}{F^2 dx^2}$, vel ſolius numeratoris $FddP - PddF$ ex illa ſubſtitutione prodiens (neglecto denominatore quadrato F^2 ſemper affirmativo) indicet Minimum Functionis propoſitae, valor negativus ipſius $FddP - PddF$ deſignet Maximum Functionis ejuſdem, & valor ipſe nulleſcens [ſubſiſtente per eandem ſubſtitutionem fluxione tertia $d^3\left(\frac{P}{Fdx^3}\right)$] nec Maximum innuat, nec Minimum. Quod ſi nulleſcente fluxione ſecunda evaneſcat ſimul etiam tertia $d^3\left(\frac{P}{Fdx^3}\right) = d^2\left(\frac{FdP - PdF}{F^2 dx^3}\right)$

$$= d.\left(\frac{FddP - PddF}{F^2dx^3} - \frac{2dF(FdP - PdF)}{F^3dx^3}\right)$$

= (ob quantitates $FdP - PdF$, & $FddP - PddF$ ex hypotheſi nihilo aequales)

$\dfrac{Fd^3P + dFd^2P - dPd^2F - Pd^3F}{F^2dx^3}$, judicium petendum erit a fluxione quarta $d^4\left(\dfrac{P}{Fdx^4}\right)$ = (ſuppoſita fluxionum praecedentium evaneſcentia)

$\dfrac{Fd^4P + 2dFd^3P - 2dPd^3F - Pd^4F}{F^2dx^4}$, quae poſitiva Minimum, negativa Maximum, nulla (dummodo ſubſiſtat fluxio ſubſequens quinta) neutrum indicabit. At ſi per ſubſtitutionem illam evaneſcat quinta quoque fluxio $d^5\left(\dfrac{P}{Fdx^5}\right)$, quae in hypotheſi nulleſcentiae fluxionum ~~antecedentium inventitur~~

$$= \frac{Fd^5P + 3dFd^4P + 2d^2Fd^3P - 2d^3Fd^2P - 3d^4FdP - Pd^5F}{F^2dx^5};$$

criterium ſuppeditabit fluxio ſubſequens ſexta

$d^6\left(\dfrac{P}{Fdx^6}\right)$, quae annihilatis antecedentibus prodit

$$=\frac{Fd^6P+4dFd^5P+5d^2Fd^4P-5d^4Fd^2P-4d^5FdP-Pd^6F}{F^2\,dx^6}.$$

Hujus ſcilicet numerator poſitivus Minimum Functionis $\frac{P}{F}$ valorem declarabit, negativus Maximum, nullus (permanente fluxione ſeptima) nec Maximum, nec Minimum ſignificabit. Atque ita porro pergendum erit, prorſus ut ſupra (15.). *Q. E. I.*

S C H O L I O N.

34. Palam eſt paullo attentius rei penetralia rimantibus, fluxionem (n+1):^{eſimam} functionis X, ſeu $\frac{P}{F}$, hoc eſt fluxionem n:^{eſimam} τȣ̃ $\frac{dX}{dx}$, ſeu ipſius $\frac{FdP-PdF}{F^2dx}$, poſito quod fluxiones omnes antecedentes annihilentur, aequalem fore ipſi $\frac{d^n FdP - d^n PdF}{F^2 dx}$.

Porro ad valorem τȣ̃, $d^n\, FdP$ obtinendum, inſtituuntur per §. 24. binae ſeries

$$d^{n+1}P,\; d^nP,\; d^{n-1}P,\; d^{n-2}P,\; d^{n-3}P,\; d^{n-4}P, \ldots\ldots dP$$

$$F,\quad dF,\quad d^2F,\quad d^3F,\quad d^4F,\quad d^5F, \ldots\ldots d^nF,$$

earumque termini eadem loca occupantes ducuntur in ſe

invicem, ac productis ſingulis adſcribuntur coefficientes Binomii Newtoniani ad poteſtatem n evecti. Ad habendum pariter valorem $-d^n\, P\, dF$ binae efformantur ſeries

$$d^n P,\ d^{n-1} P,\ d^{n-2} P,\ d^{n-3} P,\ d^{n-4} P,\ d^{n-5} P,\ \ldots\ P$$

$$dF,\quad d^2 F,\quad d^3 F,\quad d^4 F,\quad d^5 F,\quad d^6 F,\ \ldots\ d^{n+1} F,$$

ac productis terminorum eadem loca occupantium praefiguntur coefficientes memorati. Eſt itaque $d^n\, F dP$

$$= F d^{n+1} P + n\, dF d^n P + \frac{n.\,n-1.}{2} d^2 F\, d^{n-1} P$$

$$+ \frac{n.\,n-1.\,n-2.}{2.\,3} d^3 F\, d^{n-2} P + \frac{n.n-1.n-2.n-3.}{2.\,3.\,4} \times$$

$$d^4 F d^{n-3} P + \frac{n.n-1.n-2.n-3.n-4.}{2.\,3.\,4.\,5} d^5 F d^{n-4} P$$

$+$ &c. Rurſus eſt $-\, d^n\, P\, dF = -\, dF\, d^n P$

$$- n d^2 F d^{n-1} P - \frac{n.\,n-1.}{2} d^3 F\, d^{n-2} P - \frac{n.\,n-1\,n-2}{2.\,3} \times$$

$$d^{n-3} P\, d^4 F - \frac{n.\,n-1.\,n-2.\,n-3.}{2.\,3.\,4} d^{n-4} P\, d^5 F$$

$$- \frac{n.\,n-1.\,n-2.\,n-3.\,n-4.}{2.\,3.\,4.\,5} d^{n-5} P d^6 F - \&c.$$

Igitur $d^n. F\,dP - d^n. P\,dF = F d^{n+1} P \left.\begin{array}{l} +n \\ -1 \end{array}\right\} dF d^n P$

$$+\left.\begin{array}{l} \frac{n.n-1.}{2} \\ -n \end{array}\right\} d^2 F d^{n-1} P$$

$$+\left.\begin{array}{l} \frac{n.n-1.n-2.}{2.3} \\ -\frac{n.n-1.}{2} \end{array}\right\} d^3 F d^{n-2} P$$

$$+\left.\begin{array}{l} \frac{n.n-1.n-2.n-3.}{2.3.4} \\ -\frac{n.n-1.n-2.}{2.3} \end{array}\right\} d^4 F d^{n-3} P$$

$$+\left.\begin{array}{l} \frac{n.n-1.n-2.n-3.n-4.}{2.3.4.5} \\ -\frac{n.n-1.n-2.n-3.}{2.3.4} \end{array}\right\} d^5 F d^{n-4} P + \&c.$$

Facta porro terminorum reductione orietur

$$\frac{d^n. F\,dP - d^n. P\,dF}{F^2\,dx^{n+1}} = [F d^{n+1} P + .n-1. dF d^n P$$

$$+\frac{n.n-3.}{2} d^2 F d^{n-1} P + \frac{n.n-1.n-5.}{2.3} d^3 F d^{n-2} P$$

$$+\frac{n.n-1.n-2.n-7.}{2.3.4} d^4 F d^{n-3} P$$

$$+\frac{n.n-1.n-2.n-3.n-9.}{2.3.4.5}d^5 F d^{n-4} P$$

$$+\frac{n.n-1.n-2.n-3.n-4.n-11.}{2.3.4.5.6}d^6 F d^{n-5} P$$

$$+ \&c.]: F^2 d x^{n+1} = \frac{d^{n+1} X}{d x^{n+1}} = d^{n+1}.\frac{P}{F d x^{n+1}},$$

poſito tamen quod fluxiones omnes praecedentes evaneſcant. Seriem hanc terminorum ſatis concinnam & elegantem vocabimus W; ex eaque (nulla habita ratione denominatoris F^2 ſemper affirmativi) petendum erit criterium Maximorum, Minimorumque functionis propoſitae $\frac{P}{F}$. Haec nempe regula tenenda eſt: Ponatur in formula W, $n = 0$, & quantitas $FdP - dFd^0P$, ſeu $\frac{FdP - PdF}{dx}$, in quam formula abit, fiet nihilo aequalis; hujuſque quantitatis inveniantur radices; ſeu valores τȣ̃ x: Tum ſubſtituantur valores iſti pro x in formula W facto $n = 1$; ac ſi evaneſcat per ſubſtitutionem formula ipſa, ſubrogentur iidem valores in W facto $n = 2$; ſique iterum nulleſcat W, fiat eadem ſubſtitutio in W, ſumpto $n = 3$; ac ſi rurſus oriatur W nihilo aequalis, ſubſtitutio eadem fiat in W, poſito $n = 4$; ſicque perpetuo donec ea occurrat formulae

W quantitas, quae minime evaneſcat. Jam vero ſi quantitati W non evaneſcenti reſpondeat numerus $n+1$ impar, ſeu n par, caret functio $\frac{P}{F}$ Maximis, Minimiſve valoribus; ſi autem ipſi W non evaneſcenti reſpondere deprehendatur numerus $n+1$ par, ſive n impar, in Functione $\frac{P}{F}$ datur Maximum aliquod, Minimumve, primum videlicet quoties τȢ W valor oriatur negativus, alterum ubi ejuſdem W prodeat valor affirmativus.

PROBLEMA II.

35. Invenire Maxima, ac Minima functionis $\frac{PQ}{FG}$.

SOLUTIO.

Ex aequatione $\frac{dX}{dx}=\frac{FG(PdQ+QdP)-PQ(FdG+GdF)}{F^2 G^2}$ $=0$, ſive ex ſolo numeratore $=0$ eliciantur valores τȢ x, qui in functione $\frac{PQ}{FG}$ ſubrogati Maximum, Minimumve (ſi ullus erit) eidem functioni valorem inducent. Ut Maxima a Minimis diſcernantur, accipien-

da eſt fluxio ſecunda $\frac{ddX}{dx^2}$, quae (in hypotheſi fluxionis primae nulleſcentis) prodit

$$=\frac{FG(PddQ+2dPdQ+QddP)-PQ(FddG+2dFdG+GddF)}{F^2 G^2}.$$

In hac, ſeu potius in ejus numeratore (ob denominatorem ſemper affirmativum) ſubrogatus valor ipſius x quantitatem exhibebit vel affirmativam, vel negativam, vel nullam; & in primo eventu functionem $\frac{PQ}{FG}$ Minimam, in altero Maximam, in poſtremo neutram (permanente tamen fluxione tertia) indicabit. Evaneſcente autem & tertia functionis datae fluxione, confugiendum eſt ad quartam; ſicque porro pergendum, ut ante. *Q. E. I.*

S C H O L I O N.

36. Attendenti perſpectum eſt, fluxionem $(n+1)$: eſimam τ$\frac{PQ}{FG}$, ſeu n: eſimam ipſius $d.\frac{PQ}{FG}$ (poſito quod fluxiones omnes antecedentes evaneſcant) eſſe (24.)

$$\frac{d^n[FG(PdQ+QdP)]-d^n[PQ(FdG+GdF)]}{F^2G^2}$$

$=[FGd^n(PdQ+QdP)+nd.FGd^{n-1}.(PdQ+QdP)$

$+\frac{n.n-1.}{2}d^2.FGd^{n-2}(PdQ+QdP)+\frac{n.n-1.n-2.}{2.3}\times$

$d^3 FGd^{n-3}(PdQ+QdP)+\frac{n.n-1.n-2.n-3.}{2.3.4}$

$d^4 FGd^{n-4}(PdQ+QdP)+$ &c. .... $-PQ\times$

$d^n(FdG+GdF)-nd.PQd^{n-1}.(FdG+GdF)-\frac{n.n-1.}{2}\times$

$d^2.PQd^{n-2}(FdG+GdF)-\frac{n.n-1.n-2.}{2.3}d^3 PQ\times$

$d^{n-3}.(FdG+GdF)-\frac{n.n-1.n-2.n-3.}{2.3.4}d^4PQ\times$

$d^{n-4}.(FdG+GdF)-$ &c. ......$]:F^2G^2=$ (vocando A hujus quantitatis numeratorem) $\frac{A}{F^2G^2}$. Porro in magnitudine A fluxiones quaelibet ipſius FG expeditiſſime habentur per num. 24.; fluxiones autem n: eſimae, $(n-1)$: eſimae, $(n-2)$: eſimae, $(n-3)$: eſimae τ8

$(PdQ + QdP)$ ſequenti modo per ſeriem elegantiſſimam determinantur: Eſt $d^n\,(PdQ + QdP = d^n PdQ = d^n QdP$

$$= (34.)\; Pd^{n+1}Q + n\,dP\,d^nQ + \frac{n.\overline{n-1}.}{2}d^2Pd^{n-1}Q$$

$$+ \frac{n.\overline{n-1}.\overline{n-2}.}{2.3}d^3Pd^{n-2}Q + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.}{2.3.4}\times$$

$$d^4Pd^{n-3}Q + \&c. \ldots\ldots + dPd^nQ + n\,d^2Pd^{n-1}Q$$

$$+ \frac{n.\overline{n-1}.}{2}d^3Pd^{n-2}Q + \frac{n.\overline{n-1}.\overline{n-2}.}{2.3}d^4Pd^{n-3}Q$$

$$+ \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.}{2.3.4}d^5Pd^{n-4}Q + \&c. \ldots\ldots$$

$$= Pd^{n+1}Q \left.\begin{array}{l} + n \\ + 1 \end{array}\right\} dPd^nQ$$

$$\left.\begin{array}{l} + \dfrac{n.\overline{n-1}.}{2} \\ + n \end{array}\right\} d^2Pd^{n-1}Q$$

$$\left.\begin{array}{l} + \dfrac{n.\overline{n-1}.\overline{n-2}.}{2.3} \\ + \dfrac{n.\overline{n-1}.}{2} \end{array}\right\} d^3Pd^{n-2}Q$$

$$\left.\begin{array}{l} + \dfrac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.}{2.3.4} \\ + \dfrac{n.\overline{n-1}.\overline{n-2}.}{2.3} \end{array}\right\} d^4Pd^{n-3}Q$$

$+ \&c. \ldots = P d^{n+1} Q + . n+1 . dP d^n Q + . \frac{n+1 . n}{2} \times$

$d^2 P d^{n-1} Q + . \frac{n+1 . n . n-1 .}{2 . 3} d^3 P d^{n-2} Q$

$+ . \frac{n+1 . n . n-1 . n-2 .}{2 . 3 . 4} d^4 P d^{n-3} Q$

$+ . \frac{n+1 . n . n-1 . n-2 . n-3 .}{2 . 3 . 4 . 5} d^5 P d^{n-4} Q$

$+$ &c. . . . . Quae ſane quantitas praebet etiam, ut liquet, fluxiones reliquas $(n-1)$:^{eſimam}, $(n-2)$:^{eſimam}, $(n-3)$:^{eſimam},&c. ipſius $(PdQ + QdP)$. Per ſeriem omnino ſimilem inveniuntur fluxiones quantitatis $FdG + GdF$, ex quibus A coaleſcit. Jam vero ad Maxima, ac Minima functionis $\frac{PQ}{FG}$ determinanda, valores τῦ x ex aequatione $FG\,(PdQ + QdP) - PQ \times (FdG + GdF) = 0$ erutos ſubrogare oportebit loco ipſius x in formula A, ponendo ſucceſſive pro n numeros 1, 2, 3, 4, 5, &c., donec prodeat A non jam nihilo aequalis (quod perraro accidet), ſed affirmativa, vel negativa. Si itaque A tunc incipiat non evaneſcere, cum pro n ponitur numerus par; valor τῦ x

in functione $\frac{PQ}{FG}$ ſubſtitutus nullum praebet Maximum Minimumve: Si A incipiat non evaneſcere quando n capitur impar, radix x Maximum dat, Minimumve, primum prodeunte A negativo, alterum exiſtente A affirmativo; omnino ut ſupra num. 34.

PROBLEMA III.

37. Maximos, Minimoſque functionis $\frac{PQRT\ldots}{FGHL\ldots}$ valores invenire.

Solutio.

Eſt $\frac{dX}{dx} = [FGHL\ldots(PQR\ldots dT + PQT\ldots \times dR + PRT\ldots dQ + QRT\ldots dP) - PQRT\ldots \times (FGH\ldots dL + FGL\ldots dH + FHL\ldots dG + GHL\ldots dF)]: F^2 G^2 H^2 L^2\ldots = 0$ (15.), ſeu ſolus numerator $= 0$. Hujus numeratoris, quem dico N, radices loco τȣ̃ x ſubſtitutae in functione $\frac{PQRT\ldots}{FGHL\ldots}$ eam reddunt Maximam Minimamve; Maximam quidem, ſi per eandem ſubſtitutionem oriatur dN negativus; Minimam, ſi affirmativus; neutram, ſi dN fiat nullus ſubſiſtente ta-

men ddN. Evaneſcente autem & ddN, regulae ante allatae ſemper eſt inhaerendum &c. *Q.E.I.*

S C H O L I O N.

38. Evidens eſt, fluxionem (n + 1): eſimam τῦ $\frac{PQRT...}{FGHL...}$, poſito quod fluxiones ipſius praecedentes evaneſcant, fieri $= \frac{dN^n}{F^2 G^2 H^2 L^2 ...}$. Facile porro eſt per num. 24. ſeriem determinare aequalem ipſi dN^n; ita ut ad Maxima, ac Minima propoſitae functionis internoſcenda ſubſtituendi ſint in formula dN^n (denominatore $F^2 G^2 H^2 L^2 ...$ ſemper affirmativo prorſus neglecto) valores τῦ x ex aequatione $dX = 0$ eliciti ponendo ſucceſſive numeros 1, 2, 3, 4, 5, &c. in locum ipſius n quoadusque oriatur valor τῦ dN^n affirmativus, vel negativus. Si enim valor dN^n affirmativus aut negativus evadat quando n accipitur par, inita ſubſtitutio nullum propoſitae Functionis Maximum Minimumve indicabit; ſi id contingat quando n ſumitur impar, habebit data Functio Maximum aliquod, aut Minimum, primum in eventu valoris dN^n negativi, ſecundum in hypotheſi valoris ejuſdem affirmativi.

PROBLEMA IV.

39. Functionis $X = \frac{p^P}{F^f}$ Maxima, ac Minima invenire.

Solutio.

Fit de more $dX = 0 = \frac{pF^f p^{P-1} dP - f p^P F^{f-1} dF}{F^2 f}$

$= \frac{p^{P-1}(pFdP - fPdF)}{F^{f+1}}$; unde infertur $pFdP - fPdF = 0$; hujuſque aequationis radices, ſeu valores τȣ x in data functione $\frac{p^P}{F^f}$ pro x ſubſtituti functionem ipſam Maximam, vel Minimam, vel neutram efficient, prouti fluxionis ſecundae ddX valor ex eadem ſubſtitutione ortus negativus erit, vel poſitivus, vel (permanente fluxione tertia) nullus. Fluxio autem iſthaec ſecunda (ſuppoſita evaneſcentia fluxionis primae) invenitur $= \frac{p^{P-1}(pFddP + (p-f)dPdF - fPddF)}{F^{f+1}}$.

Ergo &c. *Q. E. I.*

SCHOLION I.

40. Facile captu eſt, fluxionem $(n+1)$: eſimam functionis $\frac{p^p}{Ff}$, hoc eſt n: eſimam formulae $\frac{p^{p-1}(pFdP - fPdF)}{F^{f+1}}$, poſito quod fluxiones reliquae antecedentes nihilo aequentur, fore

$$\frac{p^{p-1}(p d^n F d P - f d^n P d F)}{F^{f+1}} = (34.)\ \frac{p^{p-1}}{F^{f+1}} \times$$

$$\Big[p F d^{n+1} P + \left.\begin{matrix} np \\ -f \end{matrix}\right\} dF d^n P$$

$$+ \left.\begin{matrix} \frac{n.np-p.}{2} \\ -n f \end{matrix}\right\} d^2 F d^{n-1} P$$

$$+ \left.\begin{matrix} \frac{n.n-1.np-2p.}{2.3} \\ -\frac{n.nf-f.}{2} \end{matrix}\right\} d^3 F d^{n-2} P$$

$$+ \left.\begin{matrix} \frac{n.n-1.n-2.np-3p.}{2.3.4} \\ -\frac{n.n-1.nf-2f.}{2.3} \end{matrix}\right\} d^4 F d^{n-3} P$$

$$+\left.\begin{matrix}\dfrac{n.n-1.n-2.n-3.np-4p.}{2.3.4.5}\\ -\dfrac{n.n-1.n-2.nf-3f.}{2.3.4}\end{matrix}\right\} d^5 F d^{n-4} P + \&c....]$$

$$= \frac{P^{p-1}}{F^{f+1}} \left[p F d^{n+1} P + .np-f.\, dFd^n P \right.$$

$$+ \frac{n.np-p-2f.}{2} d^2 F d^{n-1} P + \frac{n.n-1.np-2p-3f.}{2.3} d^3 F d^{n-2} P$$

$$+ \frac{n.n-1.n-2.np-3p-4f.}{2.3.4} d^4 F d^{n-3} P$$

$$\left. + \frac{n.n-1.n-2.n-3.np-4p-5f.}{2.3.4.5} d^5 F d^{n-4} P + \&c...\right]$$

$= \frac{P^{p-1}}{F^{f+1}}.M$, nuncupato M factore altero. Hinc itaque, ut Maxima a Minimis internoſcantur, ſubſtituendi erunt valores τȣ̃ x in formula $\frac{P^{p-1}}{F^{f+1}}.M$ ponendo ſingulis vicibus pro n numeros 1, 2, 3, 4, 5, &c. uſque dum ipſius formulae valor incipiat non evaneſcere; quod quidem ſi accidat ubi pro n ſumitur numerus par, indicium eſt, nullum ex dati valoris x ſubſtitutione oriri in propoſita functione $\frac{P^p}{F^f}$ Maximum, Minimumve; ſi contra id contingat ubi n ſumitur impar, habebit praedicta functio Maximum aliquod, Minimum-

ve, Maximum prodeunte $\frac{p^{p\text{ш}1}}{F^{f+1}}$. *M* negativo, Minimum eo prodeunte affirmativo.

S C H O L I O N II.

41. Ex aequatione $\frac{p^{p\text{ш}1}\,(p\,F\,dP - f\,P\,dF)}{F^{f+1}} = 0$ colligere licet $P = 0$, cujus proinde radices, ſeu valores τȣ x in functione $\frac{P^p}{F^f}$ ſubrogati Maxima, & Minima (ſi quae fuerint) ſuppeditabunt. Ad haec autem diſtinguenda, operae pretium eſt animadvertere, fluxiones omnes ipſius $\frac{P^p}{F^f}$, quae fluxionem p: ^{eſimam} antecedunt, in hypotheſi $P = 0$ fieri nihilo aequales, fluxionemque p: ^{eſimam} ſuperſtitem eſſe

$$= \frac{p.\,p-1.\,p-2.\,p-3.\,p-4.\,p-5\ \ldots\ldots\ 1\,dP}{F^f};$$

quae ſane, ſi p fuerit numerus impar, nullum propoſitae functionis indicabit Maximum Minimumve; ſi p fuerit par, Maximum arguet, aut Minimum pro valore negativo, aut affirmativo fluxionis ejuſdem; Minimumque

ſemper designabitur, ſi uterque p, & f par fuerit, ob valorem fluxionis tunc neceſſario affirmativum.

PROBLEMA V.

42. Functionis $\frac{P^p Q^q}{F^f G^g}$ valores Maximos Minimosque determinare.

Solutio

Ex aequatione $dX = \frac{P^{p-1} Q^{q-1}(pFGQdP + qFGPdQ - fPQGdF - gPQFdG)}{F^{f+1} G^{g+1}}$ $= 0$ educantur valores τ8 x; iique in functione $\frac{P^p Q^q}{F^f G^g}$ loco ipſius x ſubſtituti Maximam illam efficient, ſi fluxio ſecunda $d^2 \frac{P^p Q^q}{F^f G^g}$ ex eadem ſubſtitutione deprehendatur negativa; Minimam, ſi fluxio iſthaec oriatur affirmativa; neutram, ſi fluxio nulleſcat, ſubſiſtente fluxione tertia; &c. ut ſupra. *Q. E. I.*

Scholion I.

43. Fluxio(n+1):^{eſima} functionis $\frac{P^p Q^q}{F^f G^g}$, ſeu fluxio n:^{eſima} formulae

$$\frac{P^{p-1}\,Q^{q-1}\,(pFGQdP + qFGPdQ - fPQGdF - gPQFdG)}{F^{f+1}\,G^{g+1}}$$

$$= \frac{P^{p-1}\,Q^{q-1}\,[QG(pFdP - fPdF) + PF(qGdQ - gQdG)]}{F^{f+1}\,G^{g+1}},$$

poſito quod fluxiones omnes antecedentes evaneſcant, invenitur $= \frac{P^{p-1}\,Q^{q-1}}{F^{f+1}\,G^{g+1}}\,[\,QG\,d^{n}(pFdP - fPdF)$

$+ nd.\,QG\,d^{n-1}(pFdP - fPdF) + \frac{n.\,n-1.}{2} \times$

$d^{2}\,QG\,d^{n-2}(pFdP - fPdF) + \frac{n.\,n-1.\,n-2.}{2.3} \times$

$d^{3}\,QG\,d^{n-3}(pFdP - fPdF) +$ &c.... $+ PF \times$

$d^{n}(qGdQ - gQdG) + nd.\,PF\,d^{n-1}(qGdQ - gQdG)$

$+ \frac{n.\,n-1.}{2}\,d^{2}\,PF\,d^{n-2}(qGdQ - gQdG)$

$+ \frac{n.\,n-1.\,n-2.}{2.3}\,d^{3}\,PF\,d^{n-3}(qGdQ - gQdG)$

$+$ &c.... $] = \frac{P^{p-1}\,Q^{q-1}}{F^{f+1}\,G^{g+1}}.\,S$, denominato S factorum altero. Per ſeries autem num. 24., & 40. inveniuntur fluxiones, quae in factore S continentur.

Ponendo igitur ſucceſſive in S loco ipſius n numeros 1, 2, 3, 4, 5, &c. quoadusque formula $\frac{P^{p-1} Q^{q-1}}{F^{f+1} G^{g+1}} . S$ affirmativa fiat, vel negativa, ſeu donec incipiat non evaneſcere, licebit, ut ante, Maxima a Minimis internoſcere. Si enim praedictae formulae non evaneſcenti, & affirmativae reſpondeat numerus n impar, propoſita functio $\frac{P^p Q^q}{F^f G^g}$ Minimum habebit; Maximum, ſi n imparem formula ſequatur negativa; neutrum, ſi formula oriatur non evaneſcens quando n par accipitur. Evidens autem eſt, poſitis p, q, f, g, imparibus, criterium hujuſmodi ex S tantummodo peti poſſe, ob valorem tunc affirmativum factoris alterius.

Scholion II.

44. Æquatio

$$\frac{P^{p-1} Q^{q-1} (pFGQdP + qFGPdQ - fPQGdF - gPQFdG)}{F^{f+1} G^{g+1}}$$

$= 0$ ſuppeditat tum $P = 0$, tum $Q = 0$. Sumpto autem $P = 0$, fluxio p: eſima functionis $\frac{P^p Q^q}{F^f G^g}$ (ob

fluxiones omnes praecedentes in hac hypotheſi evidenter nihilo aequales) inito calculo invenitur $= \frac{Q^q}{F^f G^g} \times$ $(p.p-1.p-2.p-3.p-4.p-5 \ldots 1 dP^p)$. Igitur per ſuperius demonſtrata, radix aequationis $P = 0$ in functione $\frac{P^p Q^q}{F^f G}$ ſubſtituta Maximum, vel Minimum valorem eidem impertiet quotieſcumque p fuerit par, Maximum videlicet, ſi fluxio illa p: eſima ex praedictae radicis ſubſtitutione valorem nanciſcatur negativum; Minimum, ſi fluxio ipſa valorem acquirat affirmativum: at nec Maximum, nec Minimum valorem impertiet quoties p fuerit numerus impar. Haud ſecus, accepto $Q = 0$, deducitur fluxio q: eſima functionis propoſitae, nimirum fluxio poſt praecedentes omnes evaneſcentes prima ſuperſtes $= \frac{P^p}{F^f G^g}(q.q-1.q-2. \times q-3.q-4.q-5 \ldots 1 dQ^q)$. Itaque functio $\frac{P^p Q^q}{F^f G^g}$ Maxima evadet, aut Minima ex ſubſtitutione radicis aequationis $Q = 0$ in locum τοῦ x, ſiquidem q fuerit numerus par, Maxima ſcilicet ubi fluxionis illius q: eſimae valor ex eadem ſubſtitutione prodeat negati-

vus; Minima, ubi oriatur affirmativus: neutra vero quotieſcumque impar fuerit exponens q. Ergo in univerſum, radices aequationum $P = 0$, $Q = 0$ Maxima, aut Minima ſuppeditabunt, ſi p, q, pares fuerint; neutrum, ſi impares; poſitiſque paribus non modo p, q, ſed etiam f, g, Minima tantum invenientur, ob valorem nempe fluxionum p: eſimae, q: eſimae tunc neceſſario affirmativum.

PROBLEMA VI.

45. Functionis $X = \frac{P^p\, Q^q\, R^r\, T^t \ldots}{F^f\, G^g\, H^h\, L^l \ldots}$ Maxima, ac Minima determinare.

Solutio.

Ex aequatione $dX = [F^f G^g H^h L^l \ldots (t P^p Q^q \times R^r T^{t-1} \ldots dT + r P^p Q^q T^t R^{r-1} \ldots dR + q P^p R^r T^t Q^{q-1} \ldots dQ + p Q^q R^r T^t P^{p-1} \ldots dP) - P^p \times Q^q R^r T^t \ldots (l F^f G^g H^h L^{l-1} \ldots dL + h F^f G^g \times L^l H^{h-1} \ldots dH + g F^f H^h L^l G^{g-1} \ldots dG + f G^g \times H^h L^l F^{f-1} \ldots dF)] : F^{2f} G^{2g} H^{2h} L^{2l} \ldots = P^{p-1} \times Q^{q-1} R^{r-1} T^{t-1} \ldots [PQRFGH \ldots (tLdT - lTdL)$

$+PQTFGL\ldots(rHdR - hRdH) + PRTFHL\ldots\times$ $(qGdQ - gQdG) + QRTGHL\ldots(pFdP - fPdF)]:$ $F^{f+1}G^{g+1}H^{h+1}L^{l+1}\ldots = 0$ eliciantur valores τ8 x; iique in functione X ſubſtituti valorem ipſius Maximum, Minimum, vel neutrum exhibebunt prouti ſecundae fluxionis ddX valor ex eadem ſubſtitutione proveniens negativus fuerit, affirmativus, vel permanente fluxione tertia nullus. Si & tertia evanuerit, pergendum ad quartam, &c. ut ſupra. *Q. E. I.*

Scholion I.

46. Attendenti perſpectum eſt, fluxionem $(n+1)$: eſimam functionis X, ſeu n: eſimam ipſius dX (ſuppoſita fluxionum antecedentium evaneſcentia) fore $\dfrac{P^{p-1}Q^{q-1}R^{r-1}T^{t-1}\ldots}{F^{f+1}G^{g+1}H^{h+1}L^{l+1}\ldots}$ $[PQRFGH\ldots\times$ $d^n(tLdT - lTdL) + n\,d.\,PQRFGH\ldots d^{n-1}(tLdT$ $- lTdL) + \dfrac{n.\,n-1.}{2}d^2.\,PQRFGH\ldots d^{n-2}(tLdT$ $- lTdL) + \dfrac{n.\,n-1.\,n-2.}{2.3}d^3\,PQRFGH\ldots\times$

$$d^{n+3}.(tLdT - lTdL) + \&c. \ldots\ldots + PQTFGL \ldots \times$$
$$d^{n}.(rHdR - hRdH) + n\,d.PQTFGL \ldots d^{n+1}.(rHdR$$
$$- hRdH) + \frac{n.\,n-1.}{2} d^{2}.\,PQTFGL \ldots d^{n+2}.(rHdR$$
$$- hRdH) + \frac{n.\,n-1.\,n-2.}{2.\,3} d^{3}.\,PQRFGH \ldots\ldots \times$$
$$d^{n+3}.(rHdR - hRdH) + \&c. \ldots\ldots + PRTFHL \ldots \times$$
$$d^{n}.(qGdQ - gQdG) + n\,d.PRTFHL \ldots d^{n+1}.(qGdQ$$
$$- gQdG) + \frac{n.\,n-1.}{2} d^{2}.\,PRTFHL \ldots d^{n+2}.(qGdQ$$
$$- gQdG) + \frac{n.\,n-1.\,n-2.}{2.\,3} d^{3}.\,PRTFHL \ldots\ldots \times$$
$$d^{n+3}.(qGdQ - gQdG) + \&c. \ldots\ldots + QRTGHL \ldots \times$$
$$d^{n}.(pFdP - fPdF) + n\,d.QRTGHL \ldots d^{n+1}.(pFdP$$
$$- fPdF) + \frac{n.\,n-1.}{2} d^{2}.\,QRTGHL \ldots d^{n+2}.(pFdP$$
$$- fPdF) + \frac{n.\,n-1.\,n-2.}{2.\,3} d^{3}.\,QRTGHL \ldots\ldots \times$$

$d^{n+3}.(pFdP - fPdF) + \&c. \ldots]$. In hac formula, cujus termini fluxionarii per num. 24., & 48. inveniun-

tur, facta ſubſtitutione valoris x, poſitiſque ſucceſſive numeris 1, 2, 3, 4, 5, &c. pro n, vel ea incipit non evaneſcere quando n accipitur par, vel cum n ſumitur impar. In primo eventu functio X ex ſubſtitutione dati valoris x nullum acquirit valorem Maximum, Minimumve; in altero eventu functio ipſa Minima evadit, ſi formula fuerit affirmativa; & functio oritur Maxima, ſi formula prodeat negativa. Si exponentes p, q, r, t..., f, g, h, l... impares ſupponantur, tunc quidem ob valorem factoris $\frac{P^{p+1}\, Q^{q+1}\, R^{r+1}\, T^{t+1}\ldots}{F^{f+1}\, G^{g+1}\, H^{h+1}\, T^{t+1}\ldots}$ affirmativum, ex altero formulae factore judicium petere ſufficiet.

SCHOLION II.

47. Ex aequatione $dX = 0$ palam eſt obtineri

$$1.^{\circ}\ P = 0;$$

$$2.^{\circ}\ Q = 0;$$

$$3.^{\circ}\ R = 0;$$

$$4.^{\circ}\ T = 0.$$

Jam vero aſſumpto $P = 0$, fluxio p: eſima functionis propoſitae invenitur $\frac{Q^q\, R^r\, T^t\ldots}{F^f\, G^g\, H^h\, L^l\ldots}(p.p-1.p-2.\times$

$p-3 \cdot p-4 \cdot p-5 \ldots 1\, dP^p$); reliquae omnes antecedentes evaneſcunt. Pariter accepto $Q = 0$, fluxio q: eſima ipſius X non evaneſcens praebet $\frac{P^p R^r T^t \ldots}{F^f G^g H^h L^l \ldots} \times$ $(q \cdot q-1 \cdot q-2 \cdot q-3 \cdot q-4 \cdot q-5 \ldots 1\, dQ^q)$

Rurſus poſito $R = 0$, fluxio ipſius X r: eſima, quae (reliquis evaneſcentibus) incipit non evaneſcere, oritur $\frac{P^p Q^q T^t \ldots}{F^f G^g H^h L^l \ldots}(r \cdot r-1 \cdot r-2 \cdot r-3 \cdot r-4 \cdot r-5 \ldots 1\, dR^r)$.

Denique facto $T = 0$ fluxio t: eſima ipſius X invenitur $= \frac{P^p Q^q R^r \ldots}{F^f G^g H^h L^l \ldots}(t \cdot t-1 \cdot t-2 \cdot t-3 \cdot t-4 \times$ $t-5 \ldots 1\, dT^t$). Ex his porro manifeſtum eſt, radices aequationum $P = 0$, $Q = 0$, $R = 0$, $T = 0$ in functione X ſubſtitutas Maxima, aut Minima ſuppeditare, quotieſcumque exponentes p, q, r, t numeros pares exhibuerint, & Maxima quidem, ſi fluxionum praedictarum valores per eandem ſubſtitutionem negativi prodierint; Minima, ſi affirmativi; ubi autem exponentes ipſi impares fuerint, nec Maxima, nec Minima ex radicum illarum ſubſtitutione obtineri; denique

Minimum aliquod ſemper oriri, ubi exponentes omnes *p*, *q*, *r*, *t*..., *f*, *g*, *h*, *l*... pares extiterint, ob valorem nimirum fluxionum p: ^{eſimae}, q: ^{eſimae}, &c. tunc neceſſario affirmativum.



DISQUISITIO XXI.

De Æquationibus indefinitis, deque Methodo Indeterminatarum.

1. In aequationum indefinitarum indole ac natura expendenda, earumque radicibus indagandis peculiaria quandoque ſe offerunt artificia, quae Geometrae regulis generalibus ad id praeſtandum idoneis deſtituto mirifice opitulantur, eumque praeter ſpem ad propoſitam ſibi metam perducunt.

2. Inquirebam nuper in indolem indefinitae aequationis terminorum numero utcumque magno conflatae (A) $1 + x + x^2 + x^3 + x^4 \ldots\ldots + x^n = 0$. Quum autem notas omnes regulas *fruſtra* tentaſſem, ſuccurrit tandem mihi, aequationem ipſam converti poſſe in hanc ſimpliciſſimam $x^{n+1} - 1 = 0$. Nam aequationis (A) termini ſunt continue proportionales, ut patet, eorumque ſumma ex Progreſſionum Doctrina obtinetur ſi ex facto termini poſtremi x^n in ſecundum x aufertur termini primi 1 quadratum, & reſiduum dividitur per primi & ſecundi termini differentiam;

ex quo oritur $1 + x + x^2 + x^3 \ldots\ldots + x^n = \frac{x^{n+1} - 1}{x - 1} = 0$; ac proinde (B) $x^{n+1} - 1 = 0$.

3. Id ipſum nanciſcor alia via: divido ſcilicet terminos omnes aequationis (A) per x^n, habeoque $1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} \ldots\ldots + \frac{1}{x^n} = 0$. Modo animadverto formulam $\frac{1}{x-1}$ eſſe $= \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} \ldots + \frac{1}{x^n} + \frac{1}{x^n(x-1)} = \frac{1}{x^n(x-1)} + (A) - 1$.

Quum autem ſit $(A) = 0$; oritur $\frac{1}{x-1} = \frac{1}{x^n(x-1)} - 1$, ac denique (B) $x^{n+1} - 1 = 0$, uti prius.

4. Conſidero aequationes duas

(A) $1 + x + x^2 + x^3 \ldots\ldots + x^n = 0$

(B) $x^{n+1} - 1 = 0$,

quarum haec deducta eſt ab illa, & quae iccirco praeter radicem unicam $x = 1$, alias radices omnes communes habet cum aequatione (A). Notum porro eſt, in hypotheſi n imparis, ſive $n + 1$ paris radicem alteram aequationis (B) eſſe -1: proindeque in ha

ipſa hypotheſi etiam aequatio (A) radicem realem habebit $= 1$, quod ſane manifeſtum fit quia ſingula terminorum paria evaneſcunt, & abeunt in $+1-1$.

5. Conſtat praeterea binomiae aequationis (B) radices reliquas omnes imaginarias eſſe, eaſque exprimi generaliter per $x = \cos.\frac{2h}{n+1}\pi \pm \sqrt{\left(\cos.^2\frac{2h}{n+1}\pi - 1\right)}$, denotante π ſemicircumferentiam circuli, cujus radius $= 1$, & $2h$ numeros omnes pares non majores quam $n+1$. Igitur aequationis propoſitae $1+x+x^2+x^3\ldots+x^n = 0$, exiſtente n numero impari, una tantum erit radix realis, caeteraeque imaginariae; nimirum

$$x = -1$$

$$x = \cos.\frac{2}{n+1}\pi + \sqrt{\left(\cos.^2\frac{2}{n+1}\pi - 1\right)}$$

$$x = \cos.\frac{2}{n+1}\pi - \sqrt{\left(\cos.^2\frac{2}{n+1}\pi - 1\right)}$$

$$x = \cos.\frac{4}{n+1}\pi + \sqrt{\left(\cos.^2\frac{4}{n+1}\pi - 1\right)}$$

$$x = \cos.\frac{4}{n+1}\pi - \sqrt{\left(\cos.^2\frac{4}{n+1}\pi - 1\right)}$$

. . . .

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$$x = \cos.\frac{n-1}{n+1}\pi + \sqrt{\left(\cos.^2\frac{n-1}{n+1}\pi - 1\right)}$$

$$x = \cos.\frac{n-1}{n+1}\pi - \sqrt{\left(\cos.^2\frac{n-1}{n+1}\pi - 1\right)}:$$

ſuppoſito autem *n* pari, radices omnes imaginariae ſunt, videlicet ſequentes

$$x = \cos.\frac{2}{n+1}\pi + \sqrt{\left(\cos.^2\frac{2}{n+1}\pi - 1\right)}$$

$$x = \cos.\frac{2}{n+1}\pi - \sqrt{\left(\cos.^2\frac{2}{n+1}\pi - 1\right)}$$

$$x = \cos.\frac{4}{n+1}\pi + \sqrt{\left(\cos.^2\frac{4}{n+1}\pi - 1\right)}$$

$$x = \cos.\frac{4}{n+1}\pi - \sqrt{\left(\cos.^2\frac{4}{n+1}\pi - 1\right)}$$

. . . .
. . . .
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$$x = \cos.\frac{n}{n+1}\pi + \sqrt{\left(\cos.^2\frac{n}{n+1}\pi - 1\right)}$$

$$x = \cos.\frac{n}{n+1}\pi - \sqrt{\left(\cos.^2\frac{n}{n+1}\pi - 1\right)}$$

6 Pari ratiocinandi modo demonſtrari breviſſime

poteſt Theorema, quod in Algebrae Elementis (*a*) oſtendit Eulerus methodo a Dan. Bernoullio propoſita in Actis veteribus Petropolitanae Academiae tom. III., nimirum aequationis infinitae (*C*) $x^{\infty} - x^{\infty - 1} - x^{\infty - 2}$ ..... $- x^2 - x - 1 = 0$ radicem unam realem eſſe numerum binarium. Nam ex Progreſſionum doctrina ſumma terminorum omnium hujus aequationis praeter primum eſt $\frac{1 - x^{\infty}}{x - 1}$; propterea aequatio (*C*) degenerat in $x^{\infty} + \frac{1 - x^{\infty}}{x - 1} = 0$, ſeu, in $x^{\infty + 1} - 2x^{\infty}$ $+ 1 = 0$, ſeu (facta diviſione per x^{∞}) in $x - 2$ $+ \frac{1}{x^{\infty}} = 0$, cui poſtremae aequationi evidenter ſatisfacit poſitio $x = 2$. Idem oſtendi poteſt diviſa aequatione (*C*) per x^{∞}, ex quo producitur

$$1 - \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} \ldots\ldots\ldots - \frac{1}{x^{\infty}} = 0:$$

conſtat autem ex Theoria Progreſſionum, aſſumpto

(*a*) Vid. Leonhard EULER *Vollſtändige Anleitung zur Algebra*. Zweyt. Theil, Erſt. Abſchn. Cap. 16. §. 239. Vid. etiam verſionem gallicam Jo. BERNOULLII tom. 1. §. 800.

$x = 2$ terminos poſt primum omnes $-\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3} \ldots\ldots - \frac{1}{x^\infty}$ evadere $= -1$, & conſequenter aequationi fieri ſatis.

7. Perpendo nunc aequationem (D) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 \ldots\ldots + (n+1)x = 0$, videoque oriri ipſam ex diviſione unitatis per binomium quadratum $(1-x)^2$, ita ut ſit $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 \ldots\ldots + (n+1)x^n + \frac{ax^{n+1} + bx^{n+2}}{(1-x)^2}$, ubi a, b per coefficientes Binomii determinantur. Fit itaque

(E) $bx^{n+2} + ax^{n+1} - 1 = 0$. Quare aequationis (D) reſolutio pendet a reſolutione aequationis trinomiae (E).

8. Contemplor rurſus aequationem aliam

(F) $1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 \ldots + \frac{(n+1)(n+2)x^n}{2} = 0$, hancque produci deprehendo per diviſionem unitatis factam a binomio

cubico $(1-x)^3$. Inde vero ſequens nanciſcor reſultatum

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 \ldots$$

$$+ \frac{(n+1)(n+2)x^n}{2}$$

$$+ \frac{ax^{n+1} + bx^{n+2} + cx^{n+3}}{(1-x)^3}$$; ex quo aequationem conſequor quadrinomiam (G) $cx^{n+3} + bx^{n+2} + ax^{n+1} - 1 = 0$. Quapropter aequationis indefinitae (F) reſolutio ad reſolutionem aequationis tantum quadrinomiae (G) revocatur.

9. Sit iterum indefinita aequatio

$$(H) \quad 1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 \ldots$$

$$+ \frac{(n+1)(n+2)(n+3)x^n}{2.3} = 0.$$ Oritur haec ex diviſione unitatis per binomium biquadraticum $(1-x)^4$. Diviſione autem perfecta invenitur

$$\frac{1}{(1-x)^4} = (H) + \frac{ax^{n+1} + bx^{n+2} + cx^{n+3} + dx^{n+4}}{(1-x)^4};$$

& quum ſit ex hypotheſi $(H) = 0$; ſit ideo

$$(I)\; dx^{n+4} + cx^{n+3} + bx^{n+2} + ax^{n+1} - 1 = 0.$$

Quamobrem indefinita aequatio (H) in alteram dumtaxat quinquinomiam (I) promptiſſime contrahitur.

10. Sit demum generalius aequatio indefinita

$$(K)\ 1 + nx + \frac{n(n+1)x^2}{2} + \frac{n(n+1)(n+2)x^3}{2.3}$$

$$+ \frac{n(n+1)(n+2)(n+3)x^4}{2.3.4} \ldots\ldots\ldots$$

$$+ \frac{n(n+1)(n+2)(n+3)(n+4)\ldots(n+m-1)x^m}{2.\ 3.\ 4.\ 5\ \ldots\ n}.$$

Animadverto aequationem hanc reſultare ex diviſione unitatis per binomium $1-x$ ad potentiam n evectum. Diviſione autem actu ſuſcepta conſequimur

$$\frac{1}{(1-x)^n} = (K)$$

$$+ \frac{ax^{m+1} + bx^{m+2} + cx^{m+3} + dx^{m+4} \ldots\ldots + ex^{m+n}}{(1-x)^n};$$

& quoniam eſt ex hypotheſi $(K) = 0$, prodibit demum (L) $ex^{m+n} \ldots + dx^{m+4} + cx^{m+3} + bx^{m+2} + ax^{m+1} - 1 = 0$. Propterea indefinita aequatio (K) convertibilis ſemper eſt in aliam (L) terminorum numero $n+1$ dumtaxat conſtantem.

11. Haud abſimili methodo ſummam n ter-

minorum ſeriei $\frac{1}{x}^m + \frac{2}{x^2}^m + \frac{3}{x^3}^m + \frac{4}{x^4}^m + \&c.$, deſignante m numerum quemlibet affirmativum integrum, habebimus in poteſtate. Sit enim 1.° $m = 0$, ita ut ſeries fiat $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \&c.$ Aſſumo indeterminatam magnitudinem y, & conſtituo aequalitatem $\frac{y}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4}$ &c. *in infin.* Tum duco primi aequationis membri denominatorem $x - 1$ in membrum alterum $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$ &c. *in infin.*, & productum nanciſcor $= 1$. Igitur $y = 1$. Igitur $\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$ &c. *in infin.* Quaerebatur autem ſumma tantum n terminorum: proinde oportebit ab $\frac{1}{x-1}$ ſubtrahere ſummam terminorum omnium terminum n:mum conſequentium. Jamvero termini iſti n:mum conſequentes ſunt $\frac{1}{x^{n+1}} + \frac{1}{x^{n+2}} + \frac{1}{x^{n+3}}$ &c.

in infin., ſive $\frac{1}{x^n}\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}\right.$ &c. *in infin.*

hoc eſt ex dictis $\frac{1}{x^n}\times\frac{1}{x-1}$. Conſequenter ſumma

quaeſita prodibit $=\frac{1}{x-1}-\frac{1}{x^n}\times\frac{1}{x-1}$.

12. Sit 2.° $m=1$, & ſeries $\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+\frac{4}{x^4}$ &c. *in infin.* Pono $\frac{y}{(x-1)^2}=\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+\frac{4}{x^4}$ + &c. *in infin.* Duco $(x-1)^2$ in ſeriem propoſitam, invenioque productum $=x$. Hinc colligo $y=x$, & $\frac{x}{(x-1)^2}=\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+\frac{4}{x^4}$ &c. *in infin.* Sunt autem ſeriei termini poſt n:mum $\frac{n+1}{x^{n+1}}+\frac{n+2}{x^{n+2}}+\frac{n+3}{x^{n+3}}$ $+\frac{n+4}{x^{n+4}}$ &c. *in infin.* Horum iccirco ſumma ſubtrahenda erit ab $\frac{x}{(x-1)^2}$. Dividuntur vero iſti in ſeries duas

$$\frac{n}{x^n}\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}\text{ \&c. in infin.}\right)=\frac{n}{x^n}\times\frac{1}{x-1}\ (\S.\ 11.)$$

$$\text{II. }\frac{1}{x^n}\left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}\text{ \&c. in infin.}\right)=\frac{1}{x^n}\times\frac{x}{(x-1)^2}.$$

Igitur ſumma quaeſita:

$$=\frac{x}{(x-1)^2}-\frac{1}{x^n}\times\frac{x}{(x-1)^2}-\frac{n}{x^n}\times\frac{1}{x-1}.$$

13. Sit 3°. $m=2$, adeoque ſeries $\frac{1}{x}+\frac{4}{x^2}+\frac{9}{x^3}+\frac{16}{x^4}+$ &c. *in infin.* Capio $\frac{y}{(x-1)^3}=\frac{1}{x}+\frac{4}{x^2}+\frac{9}{x^3}+\frac{16}{x^4}$ &c. *in infin.*, & ſerie ducta in $(x-1)^3$ invenio productum $=x^2+x$. Igitur $y=x^2+x$. Quare $\frac{x(x+1)}{(x-1)^3}=\frac{1}{x}+\frac{4}{x^2}+\frac{9}{x^3}+\frac{16}{x^4}$ &c. *in infin.* Jam termini ſeriei poſt n:mum ſunt $\frac{(n+1)^2}{x^{n+1}}+\frac{(n+2)^2}{x^{n+2}}+\frac{(n+3)^2}{x^{n+3}}+\frac{(n+4)^2}{x^{n+4}}$ &c. *in infin.*, vel etiam $\frac{n^2+2n+1}{x^{n+1}}+\frac{n^2+4n+4}{x^{n+2}}$

$$+\frac{n^2+6n+9}{x^{n+3}}+\frac{n^2+8n+16}{x^{n+4}}+\&c.\ in\ in-$$

fin. Hoſce porro terminos diſpeſcere licet in tres ſeries ſequentes:

$$\text{I. } \frac{n^2}{x^n}\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}\ \&c.\ in\ infin.\right)=\frac{n^2}{x^n}\times\frac{1}{x-1}.$$

$$\text{II. } \frac{2n}{x^n}\left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}+\frac{4}{x^4}\ \&c.\ in\ infin.\right)=\frac{2n}{x^n}\times\frac{x}{(x-1)^2}$$

$$\text{III. } \frac{1}{x^n}\left(\frac{1}{x}+\frac{4}{x^2}+\frac{9}{x^3}+\frac{16}{x^4}\ \&c.\ in\ infin.\right)=\frac{1}{x^n}\times\frac{x(x+1)}{(x-1)^3}.$$

Ergo ſumma quaeſita

$$=\frac{x(x+1)}{(x-1)^3}-\frac{1}{x^n}\times\frac{x(x+1)}{(x-1)^3}-\frac{2n}{x^n}\times\frac{x}{(x-1)^2}-\frac{n^2}{x^n}\times\frac{1}{x-1}.$$

14. Eſto 4° $m=3$, ſitque proinde ſeries $\frac{1}{x}+\frac{8}{x^2}+\frac{27}{x^3}+\frac{64}{x^4}+\frac{125}{x^5}+\&c.$ *in infin.* Conſtituta de more aequalitate $\frac{y}{(x-1)^4}=\frac{1}{x}+\frac{8}{x^2}+\frac{27}{x^3}$ &c. *in infin.*, & multiplicata per $(x-1)^4$ aſſequimur

$$y=(x-1)^4\left(\frac{1}{x}+\frac{8}{x^2}+\frac{27}{x^3}+\&c.\ in\ infin.\right).$$

$= x^3 + 4x^2 + x = x(x+2)^2 - 3x$. Igitur

$$\frac{x(x+2)^2 - 3x}{(x-1)^4} = \frac{1}{x} + \frac{8}{x^2} + \frac{27}{x^3} \text{ \&c. in infin.}$$

Jamvero ſeriei termini, qui $n.^{\text{mum}}$ conſequuntur, ſunt

$$\frac{(n+1)^3}{x^{n+1}} + \frac{(n+2)^3}{x^{n+2}} + \frac{(n+3)^3}{x^{n+3}} + \frac{(n+4)^3}{x^{n+4}} \text{ \&c. in}$$

infin., hique diſtribuuntur in ſeries quatuor ſequentes

$$\text{I. } \frac{n^3}{x^n}\left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} \text{ \&c. in infin.}\right) = \frac{n^3}{x^n} \times \frac{1}{x-1}$$

$$\text{II. } \frac{3n^2}{x^n}\left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3} \text{ \&c. in infin.}\right) = \frac{3n^2}{x^n} \times \frac{x}{(x-1)^2}$$

$$\text{III. } \frac{3n}{x^n}\left(\frac{1}{x} + \frac{4}{x^2} + \frac{9}{x^3} \text{ \&c. in infin.}\right) = \frac{3n}{x^n} \times \frac{x(x+1)}{(x-1)^3}$$

$$\text{IV. } \frac{1}{x^n}\left(\frac{1}{x} + \frac{8}{x^2} + \frac{27}{x^3} \text{ \&c. in infin.}\right) = \frac{1}{x^n} \times \frac{x(x+2)^2 - 3x}{(x-1)^4}.$$

Itaque demptis ab $\frac{x(x+2)^2 - 3x}{(x-1)^4}$ fit ſumma, quam

$$\text{petimus}, = \frac{x(x+2)^2 - 3x}{(x-1)^4} - \frac{1}{x^n} \times \frac{x(x+2)^2 - 3x}{(x-1)^4}$$

$$- \frac{3n}{x^n} \times \frac{x(x+1)}{(x-1)^3} - \frac{3n^2}{x^n} \times \frac{x}{(x-1)^2} - \frac{n^3}{x^n} \times \frac{1}{x-1}$$

15. Sit 5.° $m = 4$, adeoque ſeries $\frac{1}{x} + \frac{16}{x^2}$ $+ \frac{81}{x^3} + \frac{256}{x^4} +$ &c. *in infin.* $= \frac{y}{(x-1)^5}$, cujus aequalitatis multiplicato utroque membro per $(x-1)^5$ invenitur $y = x^4 + 11x^3 + 11x^2 + x$. Propterea

$$\frac{1}{x} + \frac{16}{x^2} + \frac{81}{x^3} \text{ \&c. in infin.} = \frac{x^4 + 11x^3 + 11x^2 + x}{(x-1)^5}.$$

Porro ſeriei termini, qui n.mum conſequuntur, ſunt

$$\frac{(n+1)^4}{x^{n+1}} + \frac{(n+2)^4}{x^{n+2}} + \frac{(n+3)^4}{x^{n+3}} + \frac{(n+4)^4}{x^{n+4}} + \text{\&c.}$$

in infin., nimirum $\frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{x^{n+1}}$

$$+ \frac{n^4 + 8n^3 + 24n^2 + 32n + 16}{x^{n+2}}$$

$$+ \frac{n^4 + 12n^3 + 54n^2 + 108n + 81}{x^{n+3}}$$

$$+ \frac{n^4 + 16n^3 + 96n^2 + 256n + 256}{x^{n+4}} + \text{\&c.}$$

in infin.; atque iſti conſtituunt ſeries quinque ſequentes:

$$\frac{n^4}{x^n}\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}\ \&c.\ in\ infin.\right)=\frac{n^4}{x^n}\times\frac{1}{x-1}$$

$$\text{II.}\ \frac{4n^3}{x^n}\left(\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}\ \&c.\ in\ infin.\right)=\frac{4n^3}{x^n}\times\frac{x}{(x-1)^2}$$

$$\text{III.}\ \frac{6n^2}{x^n}\left(\frac{1}{x}+\frac{4}{x^2}+\frac{9}{x^3}\ \&c.\ in\ infin.\right)=\frac{6n^2}{x^n}\times\frac{x(x+1)}{(x-1)^3}$$

$$\text{IV.}\ \frac{4n}{x^n}\left(\frac{1}{x}+\frac{8}{x^2}+\frac{27}{x^3}\ \&c.\ in\ infin.\right)=\frac{4n}{x^n}\times\frac{x(x+2)^2-3x}{(x-1)^4}$$

$$\text{V.}\ \frac{1}{x^n}\left(\frac{1}{x}+\frac{16}{x^2}+\frac{81}{x^3}\ \&c.\ in\ infin.\right)=\frac{1}{x^n}\times\frac{x^4+11x^3+11x^2+x}{(x-1)^5}.$$

Hiſce vero a praecedenti ſumma ſubtractis aſſequimur ſummam n terminorum $=\dfrac{x^4+11x^3+11x^2+x}{(x-1)^5}$

$$-\frac{1}{x^n}\times\frac{x^4+11x^3+11x^2+x}{(x-1)^5}-\frac{4n}{x^n}\times\frac{x(x+2)^2-3x}{(x-1)^4}$$

$$-\frac{6n^2}{x^n}\times\frac{x(x+1)}{(x-1)^3}-\frac{4n^3}{x^n}\times\frac{x}{(x-1)^2}-\frac{n^4}{x^n}\times\frac{1}{x-1}.$$

16. Eadem analyſi invenitur ſeries $\frac{1}{x}+\frac{2^5}{x^2}+\frac{3^5}{x^3}$ $+\frac{4^5}{x^4}+\frac{5^5}{x^5}$ &c. *in infin.* $=\dfrac{x^5+26x^4+66x^3+26x^2+x}{(x-1)^6}$.

Ex hactenus vero dictis pronum erit inferre,

1.° Summam ſeriei generalis $\frac{1}{x} + \frac{2^m}{x^2} + \frac{3^m}{x^3} + \frac{4^m}{x^4}$ &c. *in infin.* protractae exhiberi per fractionem, cujus denominator eſt binomium $x - 1$ elevatum ad poteſtatem $m + 1$, cujus poteſtatis index unitate excedit indicem poteſtatum numerorum 1, 2, 3, &c. in terminis ſeriei.

2.° Ejus fractionis numeratorem conſtare terminorum numero m.

3.° Coefficientes terminorum in ipſo numeratore poſt terminum medium eoſdem redire ac ante terminum medium, ſed ordine inverſo, inſtar unciarum Binomii.

4.° Coefficientem termini primi in eodem numeratore eſſe = 1

5.° Coefficientem ſecundi $= 2^m - \frac{m+1}{1}$.

6.° Coefficientem tertii $= 3^m - 2^m \cdot \frac{m+1}{1} + \frac{(m+1)m}{1.2}$.

7.° Coefficientem quarti $= 4^m - 3^m \cdot \frac{m+1}{1}$

$$^{m}\,.\,m\,.\,\frac{m+1}{1.2}-\frac{(m+1)\,m\,(m-1)}{1.2.3},$$

8.° Coefficientem quinti &c.

Igitur ſumma ſeriei $\frac{1^m}{x}+\frac{2^m}{x^2}+\frac{3^m}{x^3}+\frac{4^m}{x^4}$ &c. *in inf.*

$$=\frac{1}{(x-1)^{m+1}}\Big[\,x^m+\Big(2^m-\frac{m+1}{1}\Big)x^{m-1}$$

$$+\Big(3^m-2^m\times\frac{m+1}{1}+m\times\frac{m+1}{1.2}\Big)x^{m-2}$$

$$+\Big(4^m-3^m\times\frac{m+1}{1}+2^m\,m\times\frac{m+1}{1.2}$$

$$-\frac{(m+1)\,m\,(m-1)}{1.2.3}\Big)x^{m-3}+\text{\&c.}\ldots+x\,\Big].$$

Si jam ponatur

$$A=\frac{1}{x-1},$$

$$B=\frac{x}{(x-1)^2},$$

$$C=\frac{x^2+x}{(x-1)^3},$$

$$D = \frac{x^3 + 4x^2 + x}{(x-1)^4},$$

$$E = \frac{x^4 + 11x^3 + 11x^2 + x}{(x-1)^5}$$

$$F = , \text{\&c.},$$

ſumma autem ſeriei $\frac{1^m}{x} + \frac{2^m}{x^2} + \frac{3^m}{x^3}$ &c.

in infin. dicatur W, aſſequemur ſummam n tantum terminorum ipſius ſeriei ſequenti expreſſione exhibitam

$$\left(1 - \frac{1}{x^n}\right) W - \frac{n^m}{x^n} A - \frac{mn^{m-1}}{x^n} B$$

$$- \frac{m(m-1)n^{m-2}}{1.2x^n} C - \frac{m(m-1)(m-2)n^{m-3}}{1.2.3\,x^n} D$$

$$- \frac{m(m-1)(m-2)(m-3)n^{m-4}}{1.2.3.4\,x^n} E - \text{\&c.};$$

cujus ſane expreſſionis indoles manifeſta eſt, idque habet peculiare, ut quantitates complectatur A, B, C, &c. per x^n diviſas, & ſingillatim multiplicatas per termi-

nos ſingulos binomii $n-1$ ad poteſtatem m elati & explicati.

17. Omnibus Algebrae Scriptoribus familiare eſt, ex aequatione (P) $a+bx+cx^2+dx^3+ex^4\ldots+rx^n = a'+b'x+c'x^2+d'x^3+e'x^4\ldots\ldots+r'x^n$ ſequentes deducere aequalitates

$$
(M)\qquad
\begin{aligned}
a &= a' \\
b &= b' \\
c &= c' \\
d &= d' \\
e &= e' \\
&\ldots\ldots \\
&\ldots\ldots \\
&\ldots\ldots \\
r &= r'
\end{aligned}
$$

At hujuſmodi illatio doctiſſimorum licet hominum auctoritate ſuffulta fallax quandoque & captioſa eſt. Itaque operae praetium erit limites conſtituere, conditio-

neſque praeſcribere, in quibus regula aut recte pronunciat, aut fallit.

18. Igitur converto aequationem (P) in alteram (Q) $a - a' + (b - b')x + (c - c')x^2 + (d - d')x^3 \ldots + (r - r')x^n = 0$; & dico generatim, aequalitates (M) ſemper locum habere ſi quantitas x in eaquatione (P) valorem habeat diverſum a radice qualibet aequationis (Q), ita ut nulla ſit iſtius radix, quae cum magnitudine x prioris aequationis (P) conveniat: & contra, haud recte colligi, pronuncio, aequalitates (M), quotieſcumque aequatio (Q) aliquam habuerit radicem realem cum valore x prioris aequationis convenientem. Nam quum aequatio (Q) nihil aliud ſit quam aequatio ipſa (P) ſub alia forma, nequit haec verificari niſi ubi verificatur & illa. Illa autem (Q) verificatur ſemper & ſolum tum accepta x pro una qualibet ex ipſius radicibus, tum aſſumptis coefficientibus ejus ſingulis nihilo aequalibus. Igitur verificatur aequatio (P) non ſolum quando fuerit $a - a' = 0$, $b - b' = 0$, $c - c' = 0$, &c., hoc eſt quando aequalitates (M) locum habent, verum etiam quoties data qualibet inter coefficientes ſibi reſpondentes a & a', b & b', c & c', d & d', &c. inaequalitate & diſcrepantia fuerit x aequalis radici alicui reali aequationis (Q). Quapropter ex aequatio-

ne (P) inferri nequeunt aequalitates (M) niſi ubi certo conſtiterit, valorem quantitatis x in aequatione ipſa (P) ſpectatum nequaquam reperiri inter radices reales aequationis (Q).

19. Hinc ignorato utcumque valore ipſius x in aequatione (P), ſequens ſanciri poteſt

Canon I.

Si tributis quantitati x valoribus perpetuo diverſis, numeroque pluribus quam n, *aequationi (P) ſemper ſatisfit, ſtabunt aequalitates (M).*

Nam aequatio (Q) nequit plures quam n radices habere; proindeque valor aliquis ipſius x in aequatione (P) neceſſario differet a radicibus aequationis (Q).

20. A fortiori ſtatui poteſt

Canon II.

Stabunt aequalitates (M), ſi valor ipſius x in aequatione (P) variabilis fuerit, utcumque arcti ſint variationis limites.

Etenim intra hoſce limites numerus valorum ipſius x ſemper excedet n.

Huc ſpectat caſus quantitatis x infiniteſimae, vel infinitae; ſiquidem nulla eſt infiniteſima, vel infinita

quantitas in ſe determinata, & quod infiniteſimum, vel infinitum dicimus nihil aliud eſt niſi indefinitum perpetuo decreſcens, aut creſcens ultra datum quemlibet limitem.

21. Corruit ſummorum etiam Geometrarum ratiocinatio (*b*), qui in aequatione (*P*) aſſumpta $x = 0$, inde colligunt aequalitates (*M*); quum tamen ex illa aſſumptione una tantum deduci poſſit aequalitas $a = a'$, caeterae non item. Sane utcumque inaequales ſint coefficientes b & b', c & c', d & d', e & e', &c., dummodo ſit $a = a'$, ſubſiſtit aequatio (*P*), quae evaneſcentibus terminis omnibus praeter primos utriuſque membri mutatur in $a + 0 = a' + 0$, ſive $a = a'$.

22. Exemplum illegitimae illationis aequalitatum (*M*) ex aequatione (*P*) extra praeſcriptas conditiones habetur vel in tribus tantum terminis $a + bx + cx^2 = a' + b'x + c'x^2$, ubi

$$a = 1 \qquad a' = 6$$

$$b = 4 \qquad b' = 8$$

$$c = 3 \qquad c' = 2$$

$$x = 5$$

$$x = -1.$$

(*b*) Conſule unum inſtar omnium magnum EULERUM in *Intr. ad An. Inf.* tom. 1. §. 214.

Oritur enim $1 + 4x + 3x^2 = 6 + 8x + 2x^2$, & poſito $x = 5$, fit $1 + 4 \times 5 + 3 \times 25 = 6 + 8 \times 5 + 2 \times 25 = 96$, atque iterum ſumpto $x = -1$, invenitur $1 - 4 + 3 = 6 - 8 + 2 = 0$. Valores 5, & -1 quantitatis x eruti ſunt ex aequatione $(c - c')x^2 + (b - b')x + a - a' = 0$; quae de more tractata dat $x = \frac{b' - b}{2c - 2c'} \pm \sqrt{\left(\left(\frac{b' - b}{2c - 2c'}\right)^2 + \frac{a' - a}{c - c'}\right)}$.

23. Tertius ſanciri poteſt Canon de coefficientium hujuſmodi aequalitate, ſcilicet

CANON III.

Si omnes praeter unum coefficientes prioris membri aequationis (P) *aequantur coefficientibus homologis alterius membri, etiam reliquus aequatur reliquo, quicumque ſit ipſius x valor praeter nihilum.*

Sublatis enim hinc inde terminis omnibus inter ſe aequalibus, perſeverabit aequalitas inter duos illos reliquos, qui iccirco diviſi per poteſtatem, quam ibidem obtinet x, relinquent coefficientes aequales.

24. Etiamſi utrumque aequationis (P) membrum ſit nihilo aequale, non poſſunt tamen generaliter inferri aequalitates (M) quoties pro x accipitur radix aliqua membri alterutrius. Nam ex $a + bx + cx^2$

$+ dx^3 + ex^4 \ldots\ldots + rx^n = a' + b'x + c'x^2 + d'x^3 + e'x^4 \ldots + r'x^n = 0$, protinus deducitur

$$(R)\ x^n \ldots + \frac{e}{r}x^4 + \frac{d}{r}x^3 + \frac{c}{r}x^2 + \frac{b}{r}x + \frac{a}{r} = 0,$$

$$\&\ (S)\ x^n \ldots + \frac{e'}{r'}x^4 + \frac{d'}{r'}x^3 + \frac{c'}{r'}x^2 + \frac{b'}{r'}x + \frac{a'}{r'} = 0.$$

Quum autem ex Algebra notum ſit, coefficientem ſecundi termini aequationis cujuſcumque eſſe ſummam omnium radicum, coefficientem tertii ſummam productorum e binis, coefficientem quarti ſummam productorum e ternis, quinti e quaternis; &c. & coefficientem ultimi eſſe productum ex omnibus radicibus, aequales iccirco erunt aequationum (R), (S) coefficientes homologii

$$\frac{a}{r} = \frac{a'}{r'}$$

$$\frac{b}{r} = \frac{b'}{r'}$$

$$\frac{c}{r} = \frac{c'}{r'}$$

$$\frac{d}{r} = \frac{d'}{r'}$$

$$\frac{e}{r} = \frac{e'}{r'}$$

. . . . .

. . . . .

. . . . .

atque hinc eliciuntur aequalitates

$\frac{r}{r'} = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{d}{d'} = \frac{e}{e'} =$ &c., quae tantum docent, coefficientes homologos eſſe in eadem ſemper ratione, utcumque ſint inaequales.

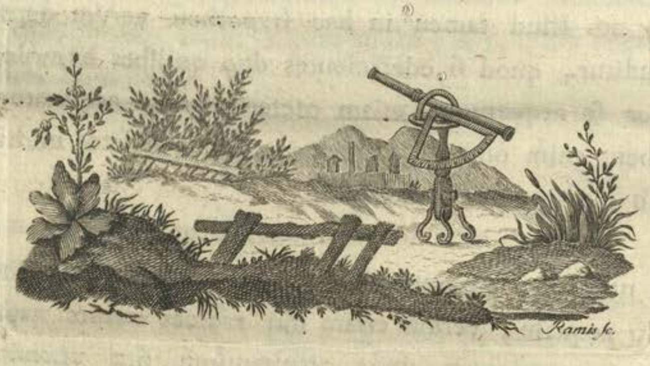
25. Id ipſum oſtenditur ducta aequatione $a + bx + cx^2 \ldots + rx^n = 0$ in m, ex quo oritur $ma + mbx + mcx^2 \ldots + mrx^n = 0$, quin tamen ſit $a = ma$, $b = mb$, &c.

26. Illud tamen in hac hypotheſi verum deprehenditur, quod ſi coefficientes duo quilibet homologi inter ſe aequantur, etiam caeteri omnes aequantur; habent enim omnes rationem eandem, ſcilicet in hoc caſu rationem aequalitatis.

27. Haec vero locum ſibi vindicant ubi non modo utrumque aequationis (P) membrum ponatur nihilo aequale, verum etiam ubi radices omnes ambarum aequationum inde reſultantium ſint utrimque communes. Si enim duae aequationes ex binis illis membris nihilo aequalibus ortae unam tantum haberent radicem communem, aut ſaltem non omnes communes, ſubrogato in utraque aequatione valore radi-

cis communis pro x, fieret utraque $= 0$, & nihilominus coefficientes utcumque inaequales eſſe poſſent, ac minime proportionales.

Satis ſint hactenus dicta ad illuſtrationem methodi per univerſam Analyſim magno emolumento uſitatae, quae vulgo dicitur Methodus Indeterminatarum, a Carteſio primum incredibili Scientiae incremento in Analyſim invecta.



DISQUISITIO XIII.

DE INFINITO LOGARITHMICO.

1. IN explicanda geneſi quantitatum logarithmicarum & exponentialium magnitudines occurrere tum infinitas tum infiniteſimas, hoc eſt aſſignabili qualibet majores minoreſve, in confeſſo eſt apud Auctores, qui hac de re accuratius deſſeruerunt, prae caeteris Magnum EULERUM in eximia *Ad Infinitorum Analyſim Introductione Cap.* VI. & VII., *Tom.* I. Verum magnitudines illas infinitas infiniteſimaſque inaudita quadam planeque admiranda donari indole ac natura, propriaſque notas & caracteres ſingularitate mirifica inſignes prae ſe ferre, id vero eſt quod Geometrarum hactenus nemo (quod ego ſciam) ſuſpicatus eſt.

2. Itaque harum quantitatum eam eſſe praecipuam notam oſtendam, ut non modo quae infinitae ſunt, infinities deprehendantur minores Infinito *primi ordinis*, quale exhibetur per infinitarum unitatum ſeriem 1 + 1 + 1 + &c. *in infin.*, ſed etiam infinito *ordinum* numero diſtent ab Infinito *ordinis primi*, utque hic infinitus *ordinum* numerus, quo infra *primi ordinis* Infini-

tum deprimuntur, ſit rurſus Infinitum diſtans infinito *ordinum* numero ab Infinito primo, atque iterum nova haec ordinum infinita diſtantia ſpectet ad infinitum, quod diſtat rurſus numero *ordinum* infinito a primo; ſicque porro ſemper, *neque* enim, magno NEWTONO ajente (*a*) *novit natura limitem*.

Sed quoniam probe intelligo, hos infinitorum ordines delicatiſſimi faſtidii hominibus ſtomachum movere, ac vel ipſum *infiniti* vocabulum ludibrio eſſe, operae pretium me facturum exiſtimo, ſi hac de re, ut nullus perverſae interpretationi relinquatur locus, quid ipſe ſentiam, diſtincte dilucideque explicavero.

3. Itaque hac ſuper re in ſententiam eo amiciſſimi doctiſſimiqne KÄSTNERI, qui in exquiſito Schediaſmate *De Vera Infiniti Notione* (*b*) rem ſuapte natura retruſam & captioſis plurium Geometrarum tendiculis implicatam ea perſpicuitate aperit enucleatque, ut nihil deſiderandum relinquat. Affero ejus locum licet paullo longiorem tum quod multa egregia & ſingularia complectitur, tum quod melius ac preſſius dici a me poſſe deſpero: » Antiquae Graeciae, inquit Cl. KÄSTNERUS, notum eſt hanc eſſe laudem, ut quae

(*a*) Princ. Lib. I. Sect. I. Lemm. XI. Schol.

(*b*) Vid. Abr. KÄSTNERI *Diſſertat. Mathemat. & Phiſic.* ALTENBURGI 1771. *Diſſert.* V.

pulchritudinis abſolutae, eadem etiam certitudinis ſummae exemplaria nobis praebeat: igitur poſtquam inter Geometras percrebuit *Infiniti* nomen, paullo poſt Cartesium, qui illud adeo adhuc reformidabat, ut mallet *Indefiniti* vocabulo uti, fuerunt qui oſtenderent, nihil illis de infinito adſertis contineri, quod non cum antiquorum placitis congrueret; alii laboris huius pertaeſi, ipſaque novitatis ſpecie delectati theoriam quandam Infiniti commenti ſunt multa continentem, quae in ſcriptis antiquorum verſatus non poſſit non, ſic ut ipſi oſtenduntur, incredulus odiſſe. Ita Fontenellius, ne ſcilicet ſeveriſſimi eruditorum Geometrae fabula mileſia plane carerent, de Infinito talem adornavit, in qua praeter alia jucundiſſima quantitates etiam reperias, quae nec finitae ſint, nec infinitae, & quibus velut *pontibus* ex finito ad infinitum eatur. Sed, ut aeque vere ac jocoſe obſervabat *B.* Hausenius, non cogitavit, ut ad hos pontes ex *finito tranſeatur*, aliis pontibus opus eſſe. Quam veram judicem *infiniti* ideam, brevibus explicare liceat.... Infinite magnam vocant quantitatem, cujus incremento limes nullus ſtatui poteſt, & quae major omni dabili concipitur; quod idem eſt ac ſi dicas ipſam non dari, alias dum omni quae datur major eſt, ſe ipſa major eſſet. Igitur *infinite magnum* non *quantitatis* nomen eſt, ſed *poſſibi-*

litatis ſine fine, *& ultra omnes limites creſcendi*. Contra *infinite parvum* non *quantitas* eſt, ſed ea *quantitatis adfectio*, qua *ſine limite decreſcere*, *omnique propoſita minor fieri poteſt*.... Hoc ſenſu infiniti vocabulo tributo, nihil paradoxi omnibus de infinito adſertis contineri perſpicitur, ut infinitum aliud alio majus immo infinities majus eſſe. Trianguli rectanguli angulus unus ſit 30. graduum: igitur crus ei oppoſitum hypothenuſae perpetuo dimidium erit, cujuſcunque magnitudinis ſint crus & hypothenuſa. Poteſt autem crus data quavis recta majus ſumi, & tunc hypothenuſa duplo ejuſdem rectae major fiet, quod dicunt crus infinitum fieri poſſe, ejuſque infiniti duplam fieri hypothenuſam. Ordinata parabolae quadrato abſciſſae exprimitur parametro pro unitate adſumpta, ut poſſint lineae ad numeros revocari: Poteſt autem abſciſſa numerum quemvis ſuperare ejuſque numeri quadratum ſuperat ordinata, quod dicitur, ordinatam exprimi quadrato abſciſſae infinitae, adeoque illam infinities continere. Ita infinita aliis infinitis infinities majora intelliguntur.... Intelligitur ſimul verum Infiniti uſum nihil continere, quod a communibus Geometrarum notionibus recedat. Qui in illo incomprehenſibilia, & tantum non contradictoria reperiunt, ambiguitate verborum plerumque falluntur, & infinitum quod revera

adfectio quantitatis eſt, pro quantitate ipſa habent. Sed quas poetis, & reliquis, qui delectare volunt, ſcriptoribus ſeverior critica interdixit dilogias, in illis ingenii gloriolam quaerere veri cuſtodes Geometras parum decebit ». His ad mentem Viri acutiſſimi, qui rem acu tetigit, praemiſſis, omnique adeo cavillandi anſa ſublata, redeo in viam.

4. Voco *Infinitum Ordinis Primi*, vel *Infinitum Primum*, vel ſimpliciter *Infinitum* illud, quod exſurgit ex ſerie unitatum ſemper continuata $1 + 1 + 1 + 1 + \&c.$ *in infin.*, vel ex continua repetitione magnitudinis cujuſcumque determinatae $a + a + a + \&c.$ *in infin.*, ſive quod repraeſentatur ab expreſſionibus $\frac{1}{1 - 1}$, $\frac{a}{1 - 1}$. Quamvis enim alterum alteri ſit inaequale, ratio tamen unius ad alterum finita eſt ac determinata, nimirum $1 : a$.

5. *Infinitum Primum* deſignabo littera n.

6. Appello *Infinitum Ordinis ſemper infiniteſimi*, vel *Infinitum Paradoxum* illud, quod non modo infinities minus eſt Infinito Primo, ſed ab eodem *diſtat*, vel infra ipſum deprimitur numero ordinum infinito, qui infinitus ordinum numerus non ſolum infinities *eſt* minor Infinito Primo, ſed rurſus ab eodem *diſtat* numero ordinum infinito; ſicque porro ſemper.

7. *Infinitum Paradoxum* deſignabo littera *p*.

8. Data aequalitate $x^x = n$, facile oſtenditur, nihil aliud eſſe poſſe x niſi *infinitum*, & quidem *infinitum paradoxum*, vel rationem habens finitam ad infinitum paradoxum p, vel denique rationem expreſſam ab alio infinito paradoxo reſpectu ipſius p, quemadmodum eſt p reſpectu infiniti primi: ſi enim *nullum novit natura limitem*, evidentiſſimum eſt, dari infinitum paradoxum p' reſpectu ipſius p, quemadmodum datur p reſpectu infiniti primi, atque hujuſmodi eſſe p'' prae p', p''' prae p'', p^{V} prae p''', ſicque porro ſine fine. Liquet id, ſi aſſumatur aequalitas $x = n$, ubi x nequit eſſe aliud niſi infinitum paradoxum ordinis quantumlibet depreſſi infra infinitum paradoxum p ordinis primi.

9. Hinc, ut infinitum paradoxum x in infinitum primum convertatur, elevandum eſt ad poteſtatem, cujus exponens eſt infinitum ipſum paradoxum.

10. Hinc radix, cujus gradus eſt *infinitum paradoxum*, educta ex infinito primo, dat infinitum ipſum paradoxum.

11. Hinc denique logarithmus infiniti paradoxi abit in logarithmum infiniti primi diviſum per infinitum ipſum paradoxum.

12. Dicatur e numerus, cujus logarithmus hyperbolicus aequatur unitati: & habetur ſequens

THEOREMA I.

$$\left(1+\frac{1}{n}\right)^n, \text{ ſeu } \left(\frac{1+n}{n}\right)^n = e.$$

Dem.

Formula $\left(\frac{n+1}{n}\right)^n$ in ſeriem ad normam Binomii Newtoniani explicata aſſequimur $\frac{1}{n^n}\Big(n^n + n.n^{n-1}$

$$+ \frac{n.n-1.}{2} n^{n-2} + \frac{n.n-1.n-2.}{2.3} n^{n-3}$$

$$+ \frac{n.n-1.n-2.n-3.}{2.3.4} n^{n-4} + \&c.\Big).$$

Sed ob $n = \infty$ evaneſcunt prae ipſo numeri 1, 2. 3, 4, &c., qui ab eodem demuntur. Igitur praedicta formula fit

$$\frac{1}{n^n}\left(n^n + n.n^{n-1} + \frac{n^2}{2}.n^{n-2} + \frac{n^3}{2.3}.n^{n-3} + \frac{n^4}{2.3.4}n^{n-4}\right.$$

$$\left. + \&c.\right) = \frac{1}{n}\left(n^n + n^n + \frac{n^n}{2} + \frac{n^n}{2.3} + \frac{n^n}{2.3.4} + \&c.\right)$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \&c.$$

Notum autem eſt, ex Logarithmorum Doctrina, eſſe

$1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \&c. = e$. Igitur

$\left(1 + \frac{1}{n}\right)^n = 2,71828183 = e$. *Q. E. D.*

13. Eſto f numerus unitate major, k vero talis, ut inter f, & k haec habeatur aequatio $f = 1 + k + \frac{k^2}{2} + \frac{k^3}{2.3} + \frac{k^4}{2.3.4} + \frac{k^5}{2.3.4.5} + \&c.$ Hoc poſito oritur

THEOREMA II.

$$f^{\frac{1}{n}} = 1 + \frac{k}{n}.$$

DEM.

Confectarium id eſt ex Capite VII. *Eulerianae Introductionis* tom. 1. nullo negotio inferendum. *Q.E.D.*

THEOREMA III.

14. $$n^{\frac{1}{n}} = 1 + \frac{p}{n}$$ *

Dem.

1.° Nequit eſſe $n^{\frac{1}{n}} = 1$, ſecus foret $n = 1^n = 1$, quod eſt abſurdum.

2.° Nequit eſſe $n^{\frac{1}{n}} = 1 + g$ (ſumpto pro g numero quolibet integro); ſecus foret $n = (1 + g)^n$: eſt autem $(1 + g)^n$ infinities major quam n, ut liquet evidentiſſime.

3.° Nequit eſſe $n^{\frac{1}{n}} = 1 + \varphi$ (deſignante φ numerum fractum quemcumque); hinc enim efficitur $n = (1 + \varphi)^n = 1 + n.\varphi + \frac{n.n-1.}{2}\varphi^2 + \frac{n.n-1.n-2.}{2.3}\varphi^3$ + &c.; quod omnino repugnat, ſiquidem ſeries $1 + n.\varphi + \frac{n.n-1.}{2}\varphi^2$ + &c. eſt manifeſte in immenſum major quam n.

* Pro p intellige vel infinitum ipſum paradoxum, de quo ſupra, vel id, quod ad infinitum paradoxum rationem habet finitam quamcumque, quodque adeo nomen idem *infiniti paradoxi* jure retinet.

4°. Non poteſt denique eſſe $n^{\frac{1}{n}} = 1 + \frac{k}{n}$ (poſito k finito); quandoquidem inde colligeretur $f^{\frac{1}{n}} = n^{\frac{1}{n}}$ §. 13., ſeu $f = n$, hoc eſt finitum aequale infinito.

Reſtat ergo, ut fiat $n^{\frac{1}{n}} = 1 + \frac{x}{n}$ exiſtente x infinito, ſed in immenſum minore quam n. Hinc autem oritur $n = \left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{n.n-1.x^2}{2n^2} + \frac{n.n-1.n-2.x^3}{2.3n^3} + \frac{n.n-1.n-2.n-3.x^4}{2.3.4n^4}$ + &c. Sed ob $n = \infty$ fit $n - 1 = n$, $n - 2 = n$, $n - 3 = n$, &c. igitur $n = \left(1 + \frac{x}{n}\right)^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \frac{x^5}{2.3.4.5}$ + &c. Inde vero conſequens eſt, quantum x non modo infinitum eſſe infinities minus quanto n, ſed ſpectare praeterea ad ordinem infinite humiliorem ordine ipſius n: nam in altero hujus aequalitatis membro aſcendit x ad poteſtatem ſemper altiorem quo longius protrahitur evolutio binomii $\left(1 + \frac{x}{n}\right)^n$, idque ſine fine. Ex hoc deni-

que facili ratiocinatione efficitur, non differre x ab *infinito paradoxo p. Q. E. D.*

15. Peccatum itaque eſt ab ingenioſiſſimo *FONTENELLIO* in Opere alioquin elegantiſſimo *De Infiniti Geometria* (a), dum affirmate pronunciat, exceſſum quantitatis ∞ ſupra unitatem non eſſe infiniteſimam quantitatem, ſed omnino *finitam*. Eſto jam

(a) Vid. *FONTENELLE Elemens de la Géomètrie de l' Infini* §. 263. Vere de hoc Opere cetera elaboratiſſimo ALEMBERTUS *in Mél. de Littér. tom. V.* §. *XV. la lecture en eſt d' autant plus dangereuſe aux jeunes Géometres, que l' Auteur y préſente ſes ſophiſmes avec une ſorte d' élégance, & pour ainſi dire, de grace, dont le ſujet ne paroiſſoit pas ſuſceptible. Il ſemble que les ouvrages géomètriques de ce Philoſophe ſoient deſtinés à produire ſur les jeunes gens qui entrent dans la carriere des Sciences, le même effet que ſes ouvrages de Belles-Lettres ſur les jeunes Littérateurs, celui d' égarer les uns & les autres par des défauts d' autant plus propres à ſéduire, qu' ils ſe trouvent & agréables par eux mêmes, & joints d' ailleurs à des beautés réelles.* Ceterum quicquid ſit de hujus Operis lectione Tironibus interdicenda, nihil aptius, nihil verius de hoc geometrico FONTENELLII libro hinc acriter vexato, inde effuſe laudato mihi dici poſſe videtur quam quod de *SENECA Inſtitut. Lib. X. Cap. I.* QUINTILIANUS, *abundat dulcibus vitiis... verum ſic quoque jam robuſtis, & ſeveriore genere ſatis firmatis legendus, vel ideo, quod exercere poteſt utrumque judicium. Multa enim probanda in eo, multa etiam admiranda ſunt: eligere modo curae ſit, quod utinam ipſe feciſſet. Digna enim fuit illa natura, quae meliora vellet, quae quod voluit effecit.*

THEOREMA IV.

Log. $n = p$.

D E M.

Per Theor. III. eſt $n^{\frac{1}{n}} = 1 + \frac{p}{n}$. Igitur ſumptis Logarithmis fiet $\frac{1}{n}$ *log.* $n =$ *log.* $\left(1 + \frac{p}{n}\right)$. Eſt autem ex Logarithmorum Doctrina *log.* $\left(1 + \frac{p}{n}\right) = \frac{p}{n} - \frac{p^2}{2n^2} + \frac{p^3}{3n^3} - \frac{p^4}{4n^4} +$ &c., haecque ſeries ob quantitatem $\frac{p}{n}$ infiniteſimam non differt ab ipſa $\frac{p}{n}$: icciroo conſequitur $\frac{1}{n}$ *log.* $n = \frac{p}{n}$, hoc eſt *log.* $n = p$. *Q. E. D.*

A L I T E R.

Ex demonſtratione Theorematis III. colligitur $n = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} +$ &c. *in infin.* Jamvero

ſeries iſthaec ex Logarithmorum Theoria numerum exprimit, cujus Logarithmus hyperbolicus eſt ipſe x, nimirum eſt $= log. \left(1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c. \right)$. Igitur $x = log. n$. Sed x non differt a p, ut patet ex Theorematis praecedentis demonſtratione. Ergo $x = p = log. n$. *Q. E. D.*

17. Convergentiae defectus in ſerie $1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} + \&c.$ demonſtrationis ſecundo loco allatae vim non infirmat eo quod ſeriei valor debet augeri ſine fine, eoque magis quo longius illa protrahitur, ac tandem adaequare infinitum ipſum primum n.

Cave ſuſpiceris, antilogiam reperiri inter hoc IV. Theorema, & §. 11., ubi logarithmus infiniti primi declaratur aequalis non infinito ſoli paradoxo, ſed facto ex hoc ipſo, & ejus logarithmo. Ibi enim *infiniti paradoxi* geneſis deducta eſt ab aequalitate $x^x = n$, quae, ut notum eſt, parit $n = 1 + x\, log. x + \frac{x^2\, log.^2 x}{2}$

$+ \frac{x^3 \, log^3. \, x}{2.3} + \frac{x^4 \, log^4. \, x}{2.3.4} +$ &c.; hic vero deducta eſt ab aequalitate $n = 1 + \grave{x} + \frac{\grave{x}^2}{2} + \frac{\grave{x}^3}{2.3}$ $+ \frac{\grave{x}^4}{2.3.4} +$ &c.; unde manifeſte colligitur $\grave{x} = x \, log. \, x$ $= log. \, n$, quod antilogiam omnem amovet. Id ipſum perſpicue infertur ex harum etiam ſerierum indole; prior enim $1 + x \, log. \, x + \frac{x^2 \, log^2. \, x}{2} + \frac{x^3 \, log^3. \, x}{2.3}$ $+ \frac{x^4 \, log^4. \, x}{2.3.4} +$ &c. numerum repraeſentat, cujus logarithmus hyperbolicus eſt ipſe $x \, log. \, x$, altera vero $1 + \grave{x} + \frac{\grave{x}^2}{2} + \frac{\grave{x}^3}{2.3} +$ &c. numerum exprimit, cujus hyperbolicus logarithmus eſt $\grave{x}$. Igitur $log. \, n = x \, log. \, x = \grave{x} = p$.

THEOREMA V.

Tab.III. Fig.13. 18. Capti in Hyperbola Apolloniana *OGK* aſymptoti partibus *AB*, *AC*, *AD*, *AE*, &c. continue proportionalibus ſine fine ita ut evadat *AS* = *infinito*

ſſimo n, erectiſque alteri aſymptoto parallelis *BG*, *CH*, *DI*, *EK*, &c. in infin.; ajo, numerum ſpatiorum inter ſe aequalium *GBCH*, *HCDI*, *IDEK.*, &c. in infin. non differre *ab infinito paradoxo p.*

D E M.

Conſtat eorum ſpatiorum inter ſe aequalium numerum, quem voco x, aequari numero terminorum progreſſionis geometricae, productae uſque ad terminum AS ſive n, dempta unitate. Eſt autem ex Progreſſionum Doctrina $\frac{AS}{AB}$, ſeu $\frac{n}{AB} = \left(\frac{AC}{AB}\right)^x$

$$= \left(\frac{AB + BC}{AB}\right)^x = \left(1 + \frac{BC}{AB}\right)^x = 1 + x \cdot \frac{BC}{AB}$$

$$+ \frac{x \cdot x - 1 \cdot}{2} \times \frac{BC^2}{AB^2} + \frac{x \cdot x - 1 \cdot x - 2 \cdot}{2 \cdot 3} \times \frac{BC^3}{AB^3}$$

$$+ \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3 \cdot}{2 \cdot 3 \cdot 4} \times \frac{BC^4}{AB^4} + \text{\&c.};$$ ac praeterea valor τῦ x nequit eſſe niſi infinitus, ut patet: Igitur ducto utroque aequalitatis membro in AB, neglectiſque numeris 1, 2, 3, &c. ab x auferendis, fiet

$$n = AB + x \cdot BC + \frac{x^2}{2} \cdot \frac{BC^2}{AB} + \frac{x^3}{2 \cdot 3} \cdot \frac{BC^3}{AB^2}$$

$$+ \frac{x^4}{2 \cdot 3 \cdot 4} \cdot \frac{BC^4}{AB^3} + \text{\&c.}$$ Quapropter traductis huc prae-

cedentibus ratiocinationibus palam eſt, valorem τ8 *x* ſpectare ad *infinitum* aliquod *paradoxum p. Q. E. D.*

COR.

Hinc conſequitur, ſpatium hyperbolicum aſymptoticum *GBSK* ſine fine protractum eſſe *infinitum paradoxum* reſpectu infinitae aſymptoti *AS* ductae in rectam quamlibet finitam. Eſto demum

THEOREMA VI.

19. In progreſſione geometrica creſcente $\div\!\div\ a,\ b,\ \frac{b^2}{a^2},\ \frac{b^3}{a^3},\ \ldots\ \frac{b^x}{a^{x-1}}$ eo uſque producta, ut terminus $\frac{b^x}{a^{x-1}}$ evadat $= n$, ajo terminorum numerum $x - 1$, adeoque & x *infinitum* eſſe *paradoxum* reſpectu ipſius n.

DEM.

Ob progreſſionem geometricam creſcentem erit $b = a + c$. Ergo $\frac{b^x}{a^{x-1}} = \frac{ab^x}{a^x} = a\left(\frac{a+c}{a}\right)^x = a\left(1+\frac{c}{a}\right)^x$

$$= a\left(1 + \frac{xc}{a} + \frac{x.\overline{x-1}.c^2}{2a^2} + \frac{x.\overline{x-1}.\overline{x-2}.c^3}{2.3\,a^3} + \frac{x.\overline{x-1}.\overline{x-2}.\overline{x-3}.c^4}{2.3.4\,a^4} + \&c.\right) = n$$

per hypotheſim. Ergo &c. *Q. E. D.*

THEOREMA VII.

20. Sint a, b numeri quilibet finiti; & $\left(\frac{a+b}{a}\right)$ $= n$: ajo, eſſe x *infinitum* aliquod *paradoxum*.

D E M.

Explicato Binomio $\left(\frac{a+b}{a}\right)^x$ in ſeriem oritur $1 + \frac{x \cdot b}{a} + \frac{x \cdot x-1 \cdot b^2}{2\, a^2} + \frac{x \cdot x-1 \cdot x-2 \cdot b^3}{2 \cdot 3\, a^3}$ $+ \frac{x \cdot x-1 \cdot x-2 \cdot x-3 \cdot b^4}{2 \cdot 3 \cdot 4\, a^4}$ + &c. in infin. $= n$. Nequit autem eſſe exponens x niſi infinitus, ut patet, ſecus aequalitas $\left(\frac{a+b}{a}\right)^x = n$ omnino repugnaret. Ergo praedicta ſeries mutabitur in $1 + \frac{x.b}{a} + \frac{x^2 . b^2}{2\, a^2}$ $+ \frac{x^3 . b^3}{2 . 3\, a^3} + \frac{x^4 . b^4}{2 . 3 . 4\, a^4}$ + &c. $= n$. Atque hinc eadem, qua ſupra, ratiocinatione efficitur, exponentem x nihil aliud eſſe poſſe niſi infinitum aliquod paradoxum. *Q. E. D.*

21. Ex poſitione $\left(\frac{a+b}{a}\right)^x = n$ conſequitur

$\left(\frac{a}{a+b}\right)^x = \frac{1}{n} =$ infiniteſimo primi ordinis. At hic poſſet quiſpiam offendere propter antilogiam, quae inopinato ſe offert: nam ſi fiat $a+b=c$, adeoque $a = c-b$, habetur $\frac{1}{n} = \left(\frac{a}{a+b}\right)^x = \left(\frac{c-b}{c}\right)^x$; & ſuſcepta evolutione binomii $\left(\frac{c-b}{c}\right)^x$ reperitur $\frac{1}{n} =$

$$1 - \frac{x.b}{c} + \frac{x.x-1.b^2}{2c^2} - \frac{x.x-1.x-2.b^3}{2.3c^3}$$

$$+ \frac{x.x-1.x-2.x-3.b^4}{2.3.4c^4} - \text{\&c. } in\ infin. = 1 -$$

$$\frac{x.b}{c} + \frac{x^2.b^2}{2c^2} - \frac{x^3.b^3}{2.3c^3} + \frac{x^4.b^4}{2.3.4c^4} - \text{\&c. } in\ infin.$$

In hac vero ſerie termini ſinguli poſt primum ſunt infinite magni, & quidem perpetuo creſcentes: conſequenter $\frac{1}{n}$, hoc eſt infiniteſimum reperitur aequale infinito; quod certe immane eſt abſurdum. At cum reputo, ſeriei terminos eſſe alternis vicibus affirmativos & negativos, proindeque alios ab aliis ita elidi poſſe, ut tota ſumma evadat infiniteſima, nullam amplius invenio antilogiam. Confirmo id exemplo ex finitis magnitudinibus petito: Evolvatur in ſeriem Binomium $(1-y)^{10}$, ut fiat $1-10y+45y^2-120y^3+$&c. Tum capiatur $y=0, 9$; eritque $(1-y)^{10}=(0, 1)^{10}=$

$1 - 9 + 45 . 0,81 - 120 . 0,729 + \&c. =$

$1 - 9 + 36 . 45 - 87,480 + \&c.$

Evidentiſſimum eſt, numerum ſatis magnum negativum, qui oritur ex collectione horum quatuor primorum [illegible] terminorum, deſtrui omnino per adjectionem ſequentium ita ut omnes collecti minimam fractionem $(0,1)^{10}$ repraeſentent.

Id ipſum Cl. Kästnerus, quem hac de re conſulebam, urbane reſpondebat Gottinga prid. Kal. Oct. an. 1777, addebatque Vir optimus, *nulla ſe offert contradictio, etſi quomodo id, quod ſeries enunciat, cum aliis veris cohaereat, paullo difficilius perſpiciatur: omnino ut in Myſteriis Religionis Chriſtianae, quae non vituperarent qui ſaeculo noſtro ſapere ſibi videntur, ſi quantum inſit myſteriorum non ſolum in rerum natura, ſed in ipſo intellectuali ſyſtemate, quo Geometria continetur, percepiſſent.*

Ceterum hujus terminorum eliſionis in ſeriebus algebraicis, in quibus ea minus apparet, luculentiſſimum mihi offert exemplum Newtoniana decantata ſeries $(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \frac{m-3n}{4n} DQ + \&c.$ Si enim educenda proponatur radix quadrata ex quadrato perfecto a^2

X

$+ 2ab + b^2$, quam radicem ſcimus eſſe $a + b$, fiatque ideo $P = a^2$, $Q = \frac{2ab + b^2}{a^2}$, $m = 1$, $n = 2$, oritur radicis quaeſitae valor expreſſus per ſeriem $a + \frac{1}{2} a \times \left(\frac{2ab + b^2}{a^2}\right) - \frac{1}{8} a \left(\frac{2ab + b^2}{a^2}\right)^2 + \frac{1}{16} a \left(\frac{2ab + b^2}{a^2}\right)^3 - \frac{5}{128} a \left(\frac{2ab + b^2}{a^2}\right)^4 +$ &c. Jamvero nemo initio cogitaverit, ſeriem hanc nullibi interruptam & in infinitum progredientem contrahi in duos tantummodo terminos $a + b$, uti res poſtulat. Attamen ſi evolvantur poteſtates $\left(\frac{2ab + b^2}{a^2}\right)^2$, $\left(\frac{2ab + b^2}{a^2}\right)^3$, $\left(\frac{2ab + b^2}{a^2}\right)^4$, &c. perſpicue apparebit, ita ſe invicem deſtruere ſeriei terminos, alios poſt alios, ut ſoli ſuperſint priores bini $a + b$.

DISQUISITIO XXV.

DE PERCUSSIONE, AUT RESISTENTIA, QUAM GLOBUS A FLUIDO IMPINGENTE, VEL IMPACTO PATITUR, PER EXPERIENTIAM DEFINIENDA.

1. Nullus huc uſque fuit in univerſa Hydrodynamica locus acrioribus Geometrarum ſtudiis diſſidiiſque vexatus quam is qui refertur ad celeberrimam illam de Percuſſione Fluidorum controverſiam, quae poſtquam per ſeptuaginta & amplius annos ſummos Viros in partes diſtraxit, noviſſime tandem per accuratiſſima Regiis ſumptibus ab illuſtrioribus Galliae Philoſophis capta pericula videtur maximam partem dirempta, & quaſi electis à tota Europa Arbitris dijudicata & compoſita (*a*). Ex hiſce demum experimen-

(*a*) Vid. *Nouvelles Expériences ſur la Réſiſtance des Fluides*, par MM. D'ALEMBERT, le Marquis DE COMDORCET, & l' Abbé BOSSUT . . . . Mr. l'Abbé BOSSUT, *Rapporteur*. Paris 1777.

tis certo conſtitit, & quaſi ſacra Legum ſanctione firmatum fuit, 1.° Reſiſtentias, quas corpus figura quali-bet praeditum, & variis velocitatibus in fluido indefinito latum patitur, eſſe quamproxime quadratis velocitatum proportionales; & hac in re Experientiam cum Theoria prope conſentire. 2.° Reſiſtentias perpendiculares ac directas, quas planae ſuperficies eadem velocitate motae patiuntur, eſſe fere ſuperficierum magnitudinibus proportionales; & hic quoque Theoriam, atque Experientiam ſatis concordes eſſe. 3.° Reſiſtentias ab obliquis motibus provenientes rationem ſequi longe diverſam a ratione duplicata ſinuum angulorum incidentiae, & conſequenter Experientiam hic magnopere diſſidere a Theoria, quae obliquam Reſiſtentiam quadrato ſinus incidentiae ponit proportionalem (*b*). 4.° Menſuram abſolutam Reſiſtentiae perpendicularis

(*b*) Lego nunc in Litteratorum Diariis, clariſſimum BOSSUT anno ſuperiore 1779. exeunte coram Pariſienſi Scientiarum Academia praelegiſſe commentariolum, in quo novis allatis experimentis, obliqui fluidorum impactus legem theoretice conſtitutam, ſed uſu mendoſam ita emendat caſtigatque, ut obliquas reſiſtentias proportionales ſtatuat non quadratis tantum ſinuum angulorum incidentiae, ut vulgaris ferebat lex, ſed proportionales ſinuum quadratis una cum poteſtatibus $3\,\frac{1}{4}$:*is* coſinuum angulorum eorundem.

ae directae, cui planum in fluido homogeneo & indefinite progrediens obijcitur, ſat proxime haberi ex pondere columnae ipſius fluidi, cujus columnae baſis aequatur plano in fluidum incurrenti, altitudo vero ea eſt, quae debetur plani velocitati, ex qua nimirum grave a quiete libere decidens velocitatem acquirit plani velocitati aequalem.

2. Everſa porro vulgari obliquorum impactuum lege de reſiſtentiae quadrato ſinus incidentiae proportionali illud conſequens erat, ut curvarum ſuperficierum reſiſtentiae per eandem legem theoretice definitae ab iis valde diſcreparent, quas experientia demonſtrat. Atque hinc reſiſtentia Globi, quam reſiſtentiae baſeos dimidiam Theoria exhibet, incerta adhuc erit minimeque explorata, donec experimentis data opera inſtitutis in lucem protracta fuerit, & citra omnem ambiguitatem conſtabilita. De experimentis dicam mox, nunc Globi reſiſtentiam theoretice, ſed via ab aliis paullulum diverſa proponam.

3. Sit itaque BCO quadrans areae circularis, iſque convertatur circa ſemidiametrum immotam BC, ita ut ex converſione areae oriatur hemiſphaerium, & ex converſione arcus quadrantalis BO producatur ſuperficies hemisphaerica. Capto jam arcu infiniteſimo Pp, ductiſque in ſemidiametrum CO perpendiculis PG, Tab. III. Fig. 12.

pg, evidens fit, ex rotatione minimi arcus Pp gigni zonam *elementarem* ſuperficiei hemisphaericae, & ex rotatione lineolae Gg gigni zonam *elementarem* circularis baſeos hemisphaerii. Liquet autem, utramque zonam ab eodem numero filamentorum fluidi ſecundum directionem PG incurrentis impelli; & conſequenter ictum contra priorem zonam ex Pp ortam ita ſe habere ad ictum contra zonam alteram genitam ex Gg quemadmodum ſe habet ſinus anguli, ſub quo percuſſio fit contra Pp ad ſinum anguli, ſub quo pulſatur Gg, ſeu ad ſinum totum: ſunt vero ſinus horum angulorum uti PG ad CO: ergo impactus obliquus in zonam ſphaericam ex Pp ortam eſt ad impactum directum in zonam circularem ex Gg productam ſicuti eſt PG ad CO. Si nunc accipiatur PG $= y$, CG $= x$, CO $= r$, & ratio diametri ad peripheriam circuli dicatur $1 : \pi$, invenitur zona circularis ex Gg orta $= 2\pi x dx$; haecque quantitas haberi jure poteſt pro menſura ictus contra zonam ipſam circularem. Ex hoc conſequitur, ictum zonae sphaericae ex Pp genitae eſſe $= \frac{2\pi y x dx}{r}$. Quamobrem integrale iſtius quantitatis $\frac{2\pi y x dx}{r}$ exprimet ictum contra ſuperficiem ſphaericam indefinitam ex rotatione arcus inde-

terminati BP productam. Invenitur porro hujuſmodi integrale, ſi ex proprietate circuli loco y ſubrogetur ipſius valor $\sqrt{(r^2 - x^2)}$; ex quo fit $\int \frac{2\pi y x dx}{r}$.

$$= \int \frac{2\pi x dx \sqrt{(r^2 - x^2)}}{r} = -\frac{2\pi}{3r}\left(r^2 - x^2\right)^{\frac{3}{2}}$$

$+$ Conſt.; cumque ictus ſit nullus evaneſcente arcu BP, vel etiam x, eruitur Conſt. $= \frac{2}{3}\pi r^2$; proindeque impactus contra ſuperficiem ſphaericam indeterminatam ex rotatione arcus BP genitam prodit $= \frac{2}{3}\pi r^2 - \frac{2\pi}{3r}\left(r^2 - x^2\right)^{\frac{3}{2}}$. Igitur ſumpto pro BP arcu quadrantali BO, vel, quod idem eſt, r pro x, oritur $\frac{2}{3}\pi r^2 =$ impactui contra ſuperficiem hemisphaericam. Et quoniam impactus contra hemisphaerii baſim, ſeu contra circulum radio CO deſcriptum exprimitur ab ipſius circuli area $= \pi r^2$, inde efficitur, percuſſionem hemisphaericae ſuperficiei aequari duabus tertiis partibus percuſſionis, quam baſis excipit.

4. Verum haec, quam dixi percuſſio, *ex toto* atque integra exercetur contra hemisphaericam ſuperficiem, hujuſque *integrae* percuſſionis pars dumtaxat eſt impulſio illa, quae in ſuperficie percuſſa motum conatur imprimere tendentem ſecundum directionem PG, vel BC fluidi impellentis. Impulſionem hanc alteram

unice ſpectarunt hactenus tum Geometrae, tum Phyſici, qui eandem vel ratione, vel experientia quaeſivere. Qua in re videtur paullulum hallucinatus vir caetera gnavus & perſpicax 's GRAVESANDUS, qui dum geometrice demonſtrat, percuſſionem hemisphaericae ſuperficiei *integram* duobus trientibus percuſſionis baſeos aequari, in hujus Theorematis confirmationem experimenta profert, quae non ad priorem *integram* percuſſionem, ſed ad alteram *partialem* referuntur, quemadmodum rem diligenter conſideranti palam fit (*c*). Ipſe quoque Cl. LECCHIUS, vir ſi quis alius conſideratus & ſagax, percuſſionem utramque confundens fidenter pronunciat, percuſſionem ſuperficiei hemisphaericae ideo GRAVESANDIO obveniſſe ſubſeſquialteram percuſſionis baſeos, non autem dimidiam ut BERNOULLIO aliiſque, quod in ſua demonſtratione pro quadrato ſinus incidentiae ſimplicem ſinum uſurpavit (*d*). Attamen re acrius excuſſa & enucleata perſpicuum fit, ubi ſuperficies duae utcumque inaequales altera oblique, altera perpendiculariter ab eodem filamentorum fluidi incurrentis numero pulſantur, percuſſionem in ſuperfi-

(*c*) Vid. *Phyſicae Elementa Mathematica* Jac. 's GRAVESANDE tom. I. Lib. III. Cap. XV.; ſed conſulenda editio quarta LUGDUNI BATAVORUM ann. 1748.

(*d*) Vid. *LECCHI Idroſtatica eſaminata ne' ſuoi Principj* Parte I. *Eſame VI.*

ciem obliquam ad percuſſionem in ſuperficiem normalem vel eſſe ſimpliciter uti eſt ſinus incidentiae ad ſinum totum, vel conjunctim uti obliqua ſuperficies ad normalem, & quadratum ſinus incidentiae ad quadratum ſinus totius: nam haec ratio compoſita ex ratione ſuperficierum, & ex ratione duplicata ſinuum incidentiae in hac hypotheſi percuſſionis ab eodem filamentorum numero factae perfecte congruit cum ratione illa ſimplici ſinuum, cui demonſtrationem ſuam Gravesandius ſuperſtruxit.

5. Porro percuſſio altera *partialis*, quae Globo motum conatur imprimere ſecundum directionem BC, vel PG fluidi impellentis, ita ex dictis expedite detegitur: Percuſſio *tota* in zonam genitam ex converſione minimi arcus Pp inventa eſt $= \frac{2\pi y x dx}{r}$; haecque urget zonam ipſam ſecundum directionem normalem PC, ſeu ſphaerae radium, ad quam directionem reducta ſemper concipitur fluidorum preſſio quaevis, cum ejus quantitas indagatur, quemadmodum Mechanici norunt. Capiatur PM $= \frac{2\pi y x dx}{r}$, eaque reſolvatur in vires duas ad ſe invicem normales MN, NP. Harum virium prior MN motum globi ſecundum PG nec adjuvat prorſus, nec impedit, eamque

praeterea elidit vis altera aequalis & oppoſita in ſecundo quadrante BA. Supereſt igitur vis ſola PN, quae globum urget juxta fluidi impellentis directionem. Quum porro ſit

$$PC : PG : : PM : PN$$

$$r : y : : \frac{2\pi y x dx}{r} : PN;$$

oritur vis $PN = \frac{2\pi y^2 x dx}{r^2}$. Hac vi tendit ſecundum PG zona ex Pp orta, & a fluido pulſata, vel eadem vi reſiſtitur juxta PG zonae ipſi fluidum impingenti. Quamobrem ſi integrale accipiatur quantitatis $\frac{2\pi y^2 x dx}{r^2}$, dabit hoc percuſſionem *partialem*, vel reſiſtentiam ſecundum PG ſuperficiei indefinitae ex rotatione arcus BP genitae. Quum vero ſit $y^2 = r^2 - x^2$, invenitur $\int \frac{2\pi y^2 x dx}{r} = \int \frac{2\pi (r^2 - x^2) x dx}{r^2}$

$= \pi x^2 - \frac{\pi x^4}{2 r^2}$ ſine ulla Conſtanti ob ſimultaneam tum percuſſionis, tum variabilis x evaneſcentiam. Itaque ſumpta $x = r$, exprimitur a quanto $\frac{1}{2}\pi r^2$ percuſſio vel reſiſtentia *partialis* ſecundum PG exercita contra ſuperficiem totam hemisphaericam, vel ſi mavis contra Globum. Haec igitur reſiſtentia dimidia eſt

ſus, quam ſuſtinet hemisphaerii baſis, ſeu circulus Globi maximus.

6. Inventa per Theoriam relatione inter Globi, & circuli maximi reſiſtentias, videndum nunc quatenus docta experimenta Hydraulicorum ad hanc rem capta Theoriae cohaereant vel adverſentur. Ad hujus argumenti illuſtrationem experimenta ſumpſere Gravesandius (*e*), & Eques De Borda (*f*).

Reſiſtentia Globi ſe habet ad reſiſtentiam
circuli maximi
Secundum experimenta Gravesandii
uti
2 : 3
Secundam Theoriam
uti
2 : 4
Secundum experimenta Equitis De Borda
uti
2 : 5.

Tanta tamque inſolens experimentorum a binis Viris ſcientiſſimis inſtitutorum diſcrepantia ſcrupulum injicit, machinamentis uſos paullo implicatioribus non po-

(*e*) Loc. cit.

(*f*) Vid. bina hujus excellentis Geometrae *De Reſiſtentia Fluidorum* Opuſcula in *Mem. de l' Acad. Roy. des Sciences de Paris ann.* 1763. & 1767.

tuiſſe effectum ſatis expurgatum ſincerumque (eſtoſ) quere, indeque germanam & accuratam reſiſtentiae aeſtimationem elicere. Suſpicio invaleſcit, ſi illarum machinarum ſtructura & compages particulatim examinetur, quidve perturbationis ex multiplici impedimentorum genere oriri debeat ſubtiliter expendatur.

7. Ad dirimendam itaque quaeſtionem adhuc fluctuantem incertamque preſto mihi ſe offert uſus Quadrantis Hydrometrici, omnium Hydraulicae inſtrumentorum ſimpliciſſimi ac paratiſſimi. Enimvero ſi in canali recto, cujus aquae curſus ſit ſatis compoſitus & horizontalis, exploretur per corpus aquae innatans certumque ſpatium certo tempore peragrans velocitas ſuperficialis; tumque Quadrantis globus filo ſuſpenſus vix intra aquae ſuperficiem demergatur, & *deviationis* angulus, quem globi filum cum verticali conſtituit, diligenter inſpiciatur, ex cognitis hujus anguli magnitudine, & velocitate aquae ſuperficiali licebit per formulam elegantem, quam mox proponam, veram reſiſtentiae globi menſuram expiſcari.

8. Ut id conſequaris, pone

Semidiametrum globi filo ſuſpenſi	$= r$
Ejus pondus extra aquam	$= p$
Pondus intra aquam	$= q$

Angulum Deviationis fili globum ſuſtinentis a directione verticali . . . . . . $= \varphi$

Celeritatem aquae ſuperficialis ſingulis [illegible] ſpatium ſingulis ſecundis ab aqua ſuperficiali decurſum . . . . . . $= c$

Spatium, quod grave libere a quiete decidens, primo temporis minuto ſecundo tranſcurrit, nimirum ped. paris. $15\frac{1}{12}$ . . . . $= g$

Ratio diametri ad peripheriam circuli . . $= 1 : \pi$

Jam vero exploratum eſt, impulſum aquae in Quadrantis globum eſſe ad globi pondus intra aquam quemadmodum eſt ſinus ad coſinum *deviationis*; hoc eſt

$cos.\ \varphi : ſin.\ \varphi :: q : \frac{q\ ſin.\ \varphi}{cos.\ \varphi}$. Eſt igitur impulſus in globum $= \frac{q\ ſin.\ \varphi}{cos.\ \varphi} = q\ tang.\ \varphi$. Porro experimenta Gallorum §. 1. indicata haud dubie decernunt ac ſanciunt, impulſum aquae in circulum globi maximum ponderi aequipollere talis aqueae calumnae, quae baſim habeat circulo maximo aequalem, altitudinemque parem illi, quae *debita* eſt aquae impellentis celeritati. Quamobrem, quoniam ex motus aequabiliter accelerati Doctrina altitudo *debita* celeritati irruentis aquae inveni-

tur $= \frac{c^2}{4g}$, circulus vero maximus $= \pi r^2$; fit columnae praedictae volumen $= \frac{\pi r^2 c^2}{4g}$. Eſt autem volumen globi $= \frac{4\pi r^3}{3}$, & paris globi aquei pondus $= p - q$, ſuntque praeterea materiae homogeneae pondera inter ſe quemadmodum volumina; habetur iccirco

$$\frac{4\pi r^3}{3} : \frac{\pi r^2 c^2}{4g} :: p - q : \frac{3c^2 (p - q)}{16gr}.$$

Ergo columnae pondus, & huic ponderi aequipollens circularis plani aquae incurſum perpendiculariter excipientis reſiſtentia oritur $= \frac{3c^2 (p - q)}{16gr}$. Dico nunc x numerum exprimentem rationem quaeſitam inter reſiſtentiam globi, & reſiſtentiam circuli maximi; habeoque analogiam

$$x : 1 :: q \text{ tang. } \varphi : \frac{3c^2 (p - q)}{16gr},$$

unde tandem elicio $x = \frac{16grq \text{ tang. } \varphi}{3c^2 (p - q)}$.

Cognitis itaque ac ſedulo definitis 1° ſemidiametro globi; 2° ejus pondere tum extra, tum intra aquam; 3° ejusdem *deviatione* a perpendiculo; 4° aquae in recto canali aequabiliter fluentis velocitate ſuperficiali, quod per corpus aquae innatans commode accurateque perficitur; invenitur ſtatim ope ſuperioris formulae ſane ſimpliciſſimae proportio illa reſiſtentiarum globi, & circuli maximi, quae diſtractos adhuc ſollicitoſque habet Hydraulicae coryphaeos. Opportune ad rem noſtram diverſo licet conſilio experientiſſimus Sueciae Phyſicus Elvius accuratione Gentis ſuae propria experimentum cepit, quod ſi ad formulam noſtram exigatur, quaeſitam proportionem luculenter patefacit (*g*). Quadranti Hydrometrico globum aptavit oſſeum, cujus diameter = 0, 17 ped. Suec., pondus extra aquam = 2397 aſſ., intra aquam = 987. Tum in fluviolo curſus aequabilis inventa ope corporis innatantis velocitate aquae ſuperficiali pedum Suec. 2, 086 ſingulis ſecundis, exploravit angulum *deviationis* globi, ſuperficie tenus demerſi, a perpendiculo, huiuſque anguli tangentem = $\frac{7}{17}$ reperit. Eſt igitur

(*g*) Vid. *Der Königl. Schwediſchen Akademie der Wiſſenſchaften Abhandlungen, auf das Jahr* 1741. *Aus dem Schwediſchen überſetzt von Abraham Gotthelf* KÄSTNER tom. III., ubi occurrit ELVII Commentariolus inſcriptus *Wie die Geschwindigkeit des Waſſers zu meſſen iſt, von. P. ELVIUS*.

$2r$	$= 0,\ 17$ *ped.*
p	$= 2397$ *aſſ.*
q	$= 987$ *aſſ.*
$p - q$	$= 1410$
g	$= 16$ *ped. Suec.*
c	$= 2,\ 086$ *ped. Suec.*
$tang.\ \varphi$	$= \frac{7}{17}$
$log.\ 2r$	$= 0,\ 2304489$
$log.\ 8$	$= 0,\ 9030900$
$log.\ q.$	$= 2,\ 9943172$
$log.\ g$	$= 1,\ 2041200$
$log.\ tang.\ \varphi$	$= 9,\ 6146491$
$log.\ 16grq\ tang.\ \varphi$	$= 3,\ 9466252$
$log.\ 3$	$= 0,\ 4771213$
$log.\ c^2$	$= 0,\ 6386286$
$log.\ (p - q)$	$= 3,\ 1492191$
$log.\ 3c^2\ (p - q)$	$= 4,\ 2649690$
$log.\ \frac{16grq\ tang.\ \varphi}{3c^2(p - q)}$	$= 9,\ 6816562$
$\frac{16grq\ tang.\ \varphi}{3c^2(p - q)}$	$= 0,\ 48046$

Habemus igitur [illegible] = 0, 48 = $\frac{1}{2}$ ſat proxime; unde conſequens fit, reſiſtentiam globi dimidiam eſſe reſiſtentiae circuli maximi; quod quidem quam pariter cum experimentis Gravesandii, & De Borda cohaeret, tam mirifice Theoriae conſonat.

10. Poſtremo quamvis in canali ad rem experiendam electo aquae curſus minime foret horizontalis, ſed horizonti inclinatus, ut plerumque uſuvenit, id tamen experimenti exitum nequaquam turbaret. Dicto enim λ inclinationis angulo, & prioribus retentis denominationibus, ex Mechanicae elementis conſequitur, eſſe actionem aquae in globum ad ejus pondus intra aquam quemadmodum eſt ſinus anguli *deviationis*, ad coſinum ſummae angulorum *deviationis*, & *inclinationis*, ſeu $cos.\,(\varphi + \lambda) : ſin.\,\varphi :: q : \frac{q\ ſin.\,\varphi}{cos.\,(\varphi + \lambda)}$.

Inventa igitur globi reſiſtentia $= \frac{q\ ſin.\,\varphi}{cos.\,(\varphi + \lambda)}$, & circuli maximi reſiſtentia $= \frac{3c^2 (p - q)}{16gr}$, ſi fiat, ut

prius, $x : y : \frac{q\ ſin.\ \varphi}{cos.(\varphi + \lambda)} : \frac{3c^2(p - q)}{16gr}$, colligitur

$$x = \frac{16grq\ ſin.\ \varphi}{3c^2(p - q)\,cos.(\varphi + \lambda)}$$; quod quaeſitam proportionem reſiſtentiarum globi, & circuli maximi in aperto ponit.



DISQUISITIO XV.

De Hora Caloris Maximi intra diem, deque Die Caloris Maximi intra annum.

1. In prima Diſquiſitione indicavimus quid Geometrae nominis celebritate clariſſimi Hallejus, & Simpsonius in tractando Problemate de Caloris Solaris menſura pro viribus praeſtiterint, & quam quiſque symbolam ad illius enodationem contulerit. Sed praeter inclyta ab illis elucubrata, & memoriae litterarum conſignata opuſcula, Mairanus rerum phyſicarum ſpeculator ingenioſus & induſtrius de hoc ipſo argumento ſemel, iterum, ac tertio diſſeruit eleganter more ſuo ac copioſe in Actis Pariſienſis Scientiarum Academiae, ubi rem phyſicae tantum indagini ſubjeciſſe contentus, Phyſicumque agens, non Phyſico-Mathematicum ab analyticis calculis vel conſulto abſtinuit, vel in illos vix ſe inſinuavit (*a*). Horum Virorum conatibus acceſſit clariſſimi Batavi Philoſophi Johannis Lu-

(a) *Mém. de l'Acad. des Sciences de Paris ann.* 1719., 1721., 1765.

LOFII induſtria, cujus elaboratum Opus Batavico idiomate exaratum, & a celeberrimo KÄSTNERO germanice verſum & auctum, id eſt *Introductio ad Mathematicam & Phyſicam Telluris Cognitionem*, Part. II. Cap. VI. multa & varia de hoc argumento complectitur cum ingenioſe diſputata, tum inventa ſubtiliter, quae & Phyſici curioſitatem explere poſſunt, & Mathematici religioni ſatisfacere (*b*). KÄSTNERUS ipſe plura hac de re habet cum in adnotationibus ad eum locum Operis Luloſiani, tum potiſſimum in *Miſcellaneis Hamburgicis* (*c*), ubi perſpicuitate ſibi adeo familiari omnia huc ſpectantia exponit & ad trutinam expendit. Noviſſime Upſalienſis Aſtronomus Fridericus MALLETUS in exquiſita opella *De Generali, ſeu Mathematica Telluris Deſcriptione* (*e*) elegantiſſimam protulit hujus Problematis ſolutionem tum in hypotheſi

(*b*) Johann LULOFS Einleitung zu der mathematiſchen und phyſikaliſchen Kenntniſs der Erdkugel aus dem Holländiſchen überſetzet von Abraham Gotthelf KÄSTNER, GÖTTINGEN und LEIPZIG 1755. in 4.° §. 575.

(*c*) HAMBURG. Magaz. II. Buch 4. St., und VIII. Buch 6. St. 5. Art. 613. St.

(*e*) Allgemeine oder Mathematiſche Beſchreibung der Erdkugel, auf Veranlaſſung der Coſmographiſchen Geſellſchaft verfaſſet von Friedrich MALLET, aus dem Schwediſchen überſetzt von LAMPERT Hinrich RÖHL, GREIFSWALD 1774. §. 62.

caloris aucti in ratione ſimplici ſinus incidentiae, tum in hypotheſi altera rationis illius duplicatae ; ſed rem delibaſſe contentus, & *caloris comparationem in reliquis terreſtribus latitudinibus praeterquam ſub Æquatore, & Polo* (ut ipſe ait) *eruere ſe poſſe* deſperans inveſtigationem vix inchoatam abrupit.

Mirum porro fuiſſet ac prorſus inſolens, ſi tot tantarumque rerum vel inventor, vel illuſtrator EULERUS, qui nullam Matheſeos partem, quam immenſa eſt, non excoluit, nullamque excoluit, quam non ditaverit vel perfecerit (*g*), quaeſtionem nobiliſſimam tantoque dignam Geometra ſilentio praeteriiſſet ; neque ſane praeteriit, nam in Actis veteribus Petropolitanae Academiae (*h*) ad rem aggreſſus ſumma, qua ſemper ſolet, ſagacitate in altiſſimam ſe conjiciens analyſim ſolutionem Problematis dedit ſi minus commodam & elegantem, exquiſitam certe & acutiſſimam.

2. Verum horum praecellentium Hominum nemo cogitavit de Problemate altero cognato & affini, utilitatiſque plurimum & ſubtilitatis habente, quo definienda proponitur hora diei, in qua aeſtivus ſolis ca-

(*g*) Ita de EULERO vere candideque pronunciat Geometra alius acerrimi ingenii CONDORCETUS tom. III. *Supplément à l'Enciclopédie* art. *Indéterminés*.

(*h*) *Comment. Acad. Petrop.* tom. XI.

lor maxime intenditur, tumque porro anni tempus aut dies, in quo prae caeteris omnibus diebus aer maxime incaleſcit. Equidem intelligo, Eulerianas formulas rite tractatas per implexas ſupputationum inextricabilium ambages huc quoque pertingere; at homo modeſtiſſimus non dubitat affirmare, *formulae iſtae tantopere ſunt compoſitae & perplexae, ut ex iis vix quicquam concludi queat*. Ad eas autem tractabiliores efficiendas & ad uſum magis idoneas cogitur Vir ſummus hypotheſes admittere a veritate rerumque natura abhorrentes: *quamobrem* (ſubdit ille) *ut aliquid ad utilitatem derivare queamus, neceſſe eſt aliquantum a veri ſimilitudine recedere*. Inde vero ad praepoſteram concluſionem perductus, quod ſane erat a minus vera hypotheſi expectandum, ſine latebris & circuitu verborum rotunde concludit, *maximus ergo calor meridianus, pariter ac maximum frigus incident poſt ſolſtitia; at diſcrimen ne unicum quidem diem adaequat, ita ut ſatis tuto ipſa ſolſtitia pro momentis, in quibus calor meridianus tum ſit maximus, tum minimus haberi queant. Quod ſi cum obſervationibus minus congruat, id hypotheſi a veritate nimium aberranti eſt tribuendum*. Haec ille, cui de accurata & perfecta Problematis ſolutione deſperanti, deque fruſtra tentata omnium analyſeos artificiorum applicatione pro-

nuncianti nemo non eandem fidem adhibeat, quam Archimedi Hieron adhiberi juſſit.

3. Quam ob rem in hac Quaeſtionis vel intractatae novitate, vel vix delibatae implicatione & incertitudine aliquid conari vireſque meas experiri volui, *nec cum fiducia inveniendi*, ut ait *Seneca* (*i*), *nec ſine ſpe*. Quum autem caſu certe magis quam ingenio in formulas inciderim valde concinnas & phaenomenis mirifice conſentaneas ſtatui rem omnem publici juris facere & doctorum judicio permittere. Eam itaque primum Problematis partem perſequar, qua definienda venit in aeſtivo quolibet die hora aeſtus Solaris maximi, partem alteram deinde aggreſſurus ubi dies quaeritur aeſtivus calori omnium maximo reſpondens.

P A R S I.

4. In triangulo ſphaerico ZPS ſit S Sol, P Polus Z Zenit, & accipiatur loci latitudo $= \lambda$, Solis declinatio $= \delta$, & ejus elevatio ſupra horizontem $= \varepsilon$. Erit igitur, ut in I. Diſquiſitione Tab. I. Fig. 1.

(*i*) *Seneca Quaeſt. Nat.* Lib. VII. Cap. 29.

$$ſin.\ PZ = cos.\ lat. = cos.\ \lambda$$

$$ſin.\ PS = cos.\ decl. = cos.\ \delta$$

$$cos.\ PZ = ſin.\ lat. = ſin.\ \lambda$$

$$cos.\ PS = ſin.\ decl. = ſin.\ \delta$$

$$cos.\ ZS = ſin.\ elev. = ſin.\ \varepsilon$$

Vulgatiſſimum autem Sphaericae Trigonomatriae Theorema ſequentem aequalitatem dat

$$cos.\ ZS = ſin.\ PZ\ ſin.\ PS\ cos.\ P + cos.\ PZ\ cos.\ PS,$$

hoc eſt facto angulo horario P, vel $SPZ = h$,

$$ſin.\ \varepsilon = cos.\ \lambda\ cos.\ \delta\ cos.\ h + ſin.\ \lambda\ ſin.\ \delta.$$

5. Jamvero aeſtus Solaris intenſio eam primum rationem ſequitur, quae ſinui altitudinis Solis ſupra horizontem directe reſpondet, veluti in I. Diſquiſitione firmavimus (*l*). Praeterea major etiam fit Solis

(*l*) Adverſus eos, qui loco ſinus altitudinis ejus quadratum uſurpant, apodictice argumentatur Vir ſummus LAMBERTUS in *Pyrometria* §. 651. *Das Quadrat*, inquit, *des Sinus der Sonnenhöhe hat hier keine Bedeutung. Es kömmt bloſs auf die Menge und Dichtigkeit der Sonnenſtralen, nicht aber auf den Stos gegen eine ebene Fläche an. Nicht die Feuertheilchen ſo aufſtoſſen und daher nothwendig wieder zurückprallen, ſondern die, welche nicht an der Fläche aufſtoſſen, ſondern in den zu erwarmenden Körper hineingehen, vermehren die Anzahl der Feuertheilchen oder die Wärme deſſelben. Dieſes iſt der Grund, warum ſchwarze Körper an der Sonne wärmer werden als weiſſe.* Quae verba latine reddita ſic ſonant: *Quadratum ſinus altitudinis Solaris nullum hic ſibi locum vin-*

Calor, quo longior eſt illius ſupra horizontem mora; augetur videlicet Calor in ratione ſimplici directa temporis, quo Sol ab horizonte ad datam altitudinem aſcendit: nam quamvis Mairanus exemplo gravium libere decidentium & ſurſum verticaliter projectorum rationem temporis hujuſmodi duplicatam ratione ſimplici potiorem habuerit, hanc tamen conſecutionem intulit ex hypotheſi minus vera, quandoquidem Calorem Solis non ſecus in aere ac tellure conſervari poſuit atque velocitas in gravibus deſcendentibus conſervatur, quae ſcilicet ſemel impreſſa nunquam amittitur (*m*).

dicat. In numero tantum & denſitate radiorum ſolarium, minime vero in illorum percuſſione vel ictu mechanico contra planum totius rei cardo vertitur. Ignearum corporis molecularum numerus, hoc eſt calor ipſius non augetur ab illis igneis particulis, quae in corporis ſuperficiem impingunt, indeque neceſſario reflectuntur, ſed ab iis tantum, quae ſuperficiem praetereuntes intra corporis meatus ſe inſinuant. Haecque cauſſa eſt cur nigra corpora Soli objecta vehementius quam alba incaleſcant.

(*m*) Praeſtat MAIRANUM ipſum diſerte more ſuo & acute hac de re diſputantem audire, quam inaudita cauſſa refellere. *La raiſon* (inquit ille) *& l'expérience concourent à prouver que la chaleur imprimée à l'air & au terrain du climat, dans un jour & à une heure quelconques, y deviendra d'autant plus grande, que le Soleil aura plus long-temps ſéjourné auparavant ſur l'horizon. La chaleur du jour Solſtical, le plus long de tous & précédé*

6. Augetur itaque Solis aeſtus dato quolibet die

des plus longs jours, dont il participe, ſera donc par-là une des principales cauſes de la ſupériorité de l'Eté ſur l'Hiver. C'eſt une ſérie croiſſante depuis le Solſtice d'Hiver juſqu'à celui d'Eté, & décroiſſante depuis le Solſtice d'Eté juſqu'à celui d'Hiver. Nous pouvons du moin la conſidérer ſous cet aſpect; car quoique le maximum *& le* minimum *n'en ſoient pas toujours à la même place, qu'ils dûſſent, en général, ſe trouver plus ou moins au-delà des Solſtices, & que la marche de la progreſſion, abſtraction faite des cauſes accidentelles, en dût devenir conſtante de l'un à l'autre terme par ſucceſſion des temps, nous ne laiſſerons pas de ſuppoſer ici le jour Solſticial le plus chaud de tous, par lui-même en raiſon de l'arc diurne ou ſemi–diurne proportionnel à ſa durée, & de plus comme précédé des jours les plus chauds; & ainſi de ſuite à l'égard de ceux-ci juſqu'au* minimum. *D'où naît une progreſſion ſemblable à celle de la force croiſſante ou décroiſſante des corps qui deſcendent ou qui montent, par l'accélération, ou par le retardement du mouvement, & que j'évaluerai auſſi de même par les quarrés des temps. Ainſi, par exemple, l'arc ſemi–diurne du jour Solſticial d'Eté à Paris, étant à peu-près double de l'arc ſemi-diurne du jour Solſticial d'Hiver, j'en conclurai l'expréſſion de cet Elément à peu-pres en raiſon de 4. à 1., pour les deux Solſtices, & ainſi de tous les autres climats, relativement à leurs latitudes & à l'arc ſemi-diurne qui y répond.*

Sans l'effet rétroactif de cette puiſſante cauſe, il feroit plus de froid, du moins dans notre hémisphère, au lever & au coucher du Soleil en Eté & le jour même du Solſtice, que dans le plus fort de l'Hiver, & réciproquement plus de chaud au lever & au coucher du Soleil en Hiver qu'en Eté; puiſque tout le reſte demeurant égal

Qui ſinus altitudinis, & ut tempus conjunctim, hoc eſt uti factum ex ſinu altitudinis Solis in tempus, quod elabitur dum Sol ab horizonte ad altitudinem illam pertingit. Quum vero aeſtus *maximus*, quem hic unicé ſpectamus, nequeat in momentum ullum antemeri-

ou nul de part & d'autre, & abſtraction faite de la chaleur imprimée les jours précédens, la diſtance du Soleil à la Terre eſt plus grande en Eté qu' en Hiver d'environ un million de lieues. D'où il ſuit, que le calcul de notre Eté & de notre Hiver, ſeroit défectueux, ſi nous ne faiſions réſulter ce quatrième Elément que du ſimple arc ſemi-diurne. Mem. de l' Acad. des Sciences de Paris pour l'année 1765. Ecquis vero MAIRANO concedat, (quod ratiocinationis ſuae fundamentum eſt) ſolſtitialem diem eſſe omnium aeſtivorum calidiſſimum? Ecquis det, caloris incrementa progreſſionem conſtituere illi ſimilem, quam incrementa velocitatis in gravibus libere decidentibus componunt? Ecquis denique MAIRANO aſſentiatur, conſervari calorem in corporibus perinde ac velocitas lapſu acquiſita conſervatur, quam ſcilicet ſemel adeptam labentia corpora ſublatis obſtaculis continenter tuentur? Quam parum haec cum ſeverioribus Phyſicorum dictatis cohaereant, nemo eſt qui non ſentiat. Hac fortaſſe de cauſſa Philoſophus elegantiſſimus perpetuuſque MAIRANI admirator BAILLY caloris aeſtimationem exponens, & elementum illud exagitans, quod a tempore vel arcu ſemidiurno conſtituitur, dum officioſis verbis duci blanditur ſuo, re deſciſcit ab ipſo, & repudiato arcus ſemidiurni quadrato ſimplicem arcum uſurpat. Vid. BAILLY *Lettres ſur l' Origine des Sciences* Lett. IX.

dianum incidere, ut per ſe patet (*n*), proportionalis, idcirco caloris quantitas pro momento quovis pomeridiano invenietur, ſi ducatur altitudo Solis pomeridiana in arcum vel angulum ſemidiurnum diei propoſitae auctum angulo horario poſt meridiem. Dicatur itaque a angulus vel arcus ſemidiurnus, ſitque SPZ, vel h horarius angulus poſt meridiem; erit $a + h$ menſura temporis ab ortu ſolis ad momentum illud uſque pomeridianum. Quapropter caloris intenſio proportionalis erit analyticae expreſſioni $(a + h)\ \mathit{ſin.}\ \varepsilon$, ſive ob aequalitatem antea conſtitutam

$$\mathit{ſin.}\ \varepsilon = \cos.\lambda \cos.\delta \cos.h + \mathit{ſin.}\lambda\ \mathit{ſin.}\delta,$$

caloris relativi formula pro momento quovis pomeridiano diei cujuſcumque haec habebitur

$$(a + h)(\cos.\lambda \cos.\delta \cos.h + \mathit{ſin.}\lambda\ \mathit{ſin.}\delta)$$
$$= a \cos.\lambda \cos.\delta \cos.h + a\ \mathit{ſin.}\lambda\ \mathit{ſin.}\delta + h \cos.\lambda \times \cos.\delta \cos.h + h\ \mathit{ſin.}\lambda\ \mathit{ſin.}\delta.$$

Igitur hujus formulae differentiale, accepta h variabili, inventum, & nihilo aequatum functionem ipſius

(*n*) Ex vulgatiſſima Inertiae lege fit, ut qui a continenti cauſſae alicujus actione gignitur effectus, maximus non evadat niſi ſat diu poſt momentum illud, quo maxima eſt cauſſae actio: tunc enim ſolummodo effectus exiſtit omnium maximus, cum decrementum ab imminuta cauſſae actione profectum proxime par eſt incremento, quod ipſe capit ex actione continuata.

praebebit, quae pomeridianum aeſtus maximi momentum patefaciet. Propterea erit

$$-a\,dh\,\text{ſin.}\,h\,\cos.\lambda\,\cos.\delta + dh\,\cos.h\,\cos.\lambda\,\cos.\delta$$
$$-h\,dh\,\text{ſin.}\,h\,\cos.\lambda\,\cos.\delta + dh\,\text{ſin.}\,\lambda\,\text{ſin.}\,\delta = 0, \text{ vel}$$

$$(M) \qquad \cos.h - a\,\text{ſin.}\,h - h\,\text{ſin.}\,h + \frac{\text{ſin.}\,\lambda\,\text{ſin.}\,\delta}{\cos.\lambda\,\cos.\delta} = 0.$$

7. Ut modo ex hac aequatione tranſcendenti quaeſitam eliciam quantitatis h menſuram, experior primo Analyſtis familiariſſimam ſerierum reverſionem.

Ex functionum circularium doctrina conſtat, eſſe

$$\cos.\,h = 1 - \frac{h^2}{2} + \frac{h^4}{2.3.4} - \&c.$$

$$\text{ſin.}\,h = h - \frac{h^3}{2.3} + \frac{h^5}{2.3.4.5} - \&c.$$

Hiſce valoribus in aequatione (M) ſubrogatis oritur

$$1 - ah - \frac{3}{2}h^2 + \frac{a}{6}h^3 + \frac{5}{24}h^4 \ldots + \frac{\text{ſin.}\,\lambda\,\text{ſin.}\,\delta}{\cos.\lambda\,\cos.\delta} = 0;$$

deinde diviſione per $-a$ inſtituta prodit infinita aequatio

$$h + \frac{3}{2a}h^2 - \frac{1}{6}h^3 - \frac{5}{24a}h^4 \ldots - \frac{\text{ſin.}\lambda\,\text{ſin.}\delta + \cos.\lambda\,\cos.\delta}{a\cos.\lambda\,\cos.\delta} = 0, \text{ ſeu}$$

$$h + \frac{3}{2a}h^2 - \frac{1}{6}h^3 - \frac{5}{24a}h^4 \ldots\ldots\ldots - r = 0,$$

poſito videlicet $r = \frac{\text{ſin.}\,\lambda\;\text{ſin.}\,\delta + \cos.\,\lambda\;\cos.\,\delta}{a\cos.\,\lambda\;\cos.\,\delta}$.

8. Pergo modo, aſſumoque

$$\begin{array}{lll}
h & = Ar + Br^2 + Cr^3 + Dr^4 + \&c. & \\
+\frac{3}{2a}\,h^2 & = & +\frac{3}{2a}A^2r^2 + \frac{3}{a}ABr^3 + \frac{3}{2a}B^2r^4 \\
 & & +\frac{3}{a}ACr^4 \\
-\frac{1}{6}\,h^3 & = & -\frac{1}{6}A^3r^3 - \frac{1}{2}A^2Br^4 \\
-\frac{5}{24a}\,h^4 & = & -\frac{5}{24a}A^4r^4 \\
-r & = -r &
\end{array}$$

Ex comparatione coefficientium indeterminatorum ſequentes oriuntur aequalitates

$$Ar - r = 0$$

$$Br^2 + \frac{3}{2a}A^2r^2 = 0$$

$$Cr^3 + \frac{3}{a}ABr^3 - \frac{1}{6}A^3r^3 = 0$$

$$Dr^4 + \frac{3}{2a}B^2r^4 - \frac{1}{2}A^2Br^4 + \frac{3}{a}ACr^4 - \frac{5}{24a}A^4r^4 = 0.$$

Inde vero iſti prodeunt coefficientium valores

$$A = 1$$

$$B = -\frac{3}{2a}$$

$$C = \frac{1}{6} + \frac{9}{2a^2}$$

$$D = -\frac{25}{24a} - \frac{135}{8a^3}.$$

Igitur

$$h = r - \frac{3}{2a}r^2 + \left(\frac{1}{6} + \frac{9}{2a^2}\right)r^3$$

$$-\left(\frac{25}{24a} + \frac{135}{8a^3}\right)r^4 \text{ \&c.}$$

9. Pergo porro, & numericam temporis h menſuram hinc conor elicere ad hunc modum.

Ticini Regii diebus aeſtivis ſolſtitialibus tempus ſemidiurnum, ſive ab ortu Solis uſque ad meridiem invenitur $= 7^h\, 56^m$, quod in arcum ſemidiurnum aequatoris converſum praebet arcum ipſum $= 2,077$, quum tota peripheria, quae horis 24. reſpondet, ſit $= 6,283$ ex nota diametri ad circumferentiam ratione. Quare

	$= 2, 077$
λ	$= 45^\circ\ 11'$
	$= 23^\circ\ 28'$
log. ſin. λ	$= 9, 8508702$
log. ſin. δ	$= 9, 6001181$
log. ſin. λ *ſin.* δ	$= 9, 4509883$
ſin. λ *ſin.* δ	$= 0, 2825$
log. cos. λ	$= 9, 8480909$
log. cos. δ	$= 9, 9625076$
log. cos. λ *cos.* δ	$= 9, 8105985$
cos. λ *cos.* δ	$= 0, 6465$
ſin. λ *ſin.* δ + *cos.* λ *cos.* δ	$= 0, 9290$
log. a	$= 0, 3174116$
log. a cos. λ *cos.* δ	$= 0, 1280101$
log. (*ſin.* λ *ſin.* δ + *cos.* λ *cos.* δ)	$= 9, 9680157$
log. $\dfrac{ſin.\lambda\ ſin.\delta + cos.\lambda\ cos.\delta}{a\ cos.\lambda\ cos.\delta} = log. r$	$= 9, 8400056$
r	$= 0, 6918$
log. r^2	$= 9, 6800112$
log. 3	$= 0, 4771213$

$log.\ 3r^2$	. . . .	$= 0,\ 1571325$
$log.\ 2$	. . . .	$= 0,\ 3010300$
$log.\ a$	. . . .	$= 0,\ 3174116$
$log.\ 2a$	. . . .	$= 0,\ 6184416$
$log.\ \frac{3r^2}{2a}$	. . . .	$= 9,\ 5386909$
$-\ \frac{3r^2}{2a}$	. . . .	$= -0,\ 3457$
$log.\ a^2$	. . . .	$= 0,\ 6348232$
$log.\ 2$	. . . .	$= 0,\ 3010300$
$log.\ 2a^2$	. . . .	$= 0,\ 9358532$
$log.\ 9$	. . . .	$= 0,\ 9542426$
$log.\ \frac{9}{2a^2}$	. . . .	$= 0,\ 0183894$
$\frac{9}{2a^2}$	. . . .	$= 1,\ 043$
$\frac{1}{6}$	. . . .	$= 0,\ 167$
$\frac{1}{6} + \frac{9}{2a^2}$	. . . .	$= 1,\ 210$

$log.\left(\frac{1}{6}+\frac{9}{2a^2}\right)$	=	0, 0827854
$log.\ r^3$	=	9, 5200168
$log.\left(\frac{1}{6}+\frac{9}{2a^2}\right)r^3$	=	9, 6028022
$\left(\frac{1}{6}+\frac{9}{2a^2}\right)r^3$	=	0, 4007
$log.\ 25$	=	1, 3979400
$log.\ 24$	=	1, 3802112
$log.\ a$	=	0, 3174116
$log.\ 24\,a$	=	1, 6976228
$log.\ \frac{25}{24a}$	=	9, 7003172
$\frac{25}{24\,a}$	=	0, 5016
$log.\ 135$	=	2, 1303338
$log.\ (2a)^3 = log.\ 8a^3$	=	1, 8553248
$log.\ \frac{135}{8a^3}$	=	0, 2750090
$\frac{135}{8a^3}$	=	1, 884

$\frac{25}{24\,a} + \frac{135}{8\,a^3}$ . . . . $=$ 2, 3856

$log. \left(\frac{25}{24\,a} + \frac{135}{8\,a^3} \right)$ . . . . . $=$ 0, 3775976

$log.\ r^4$ . . . . $=$ 9, 3600224

$log. \left(\frac{25}{24\,a} + \frac{135}{8\,a^3} \right) r^4$ . . . . $=$ 9, 7376200

$- \left(\frac{25}{24\,a} + \frac{135}{8\,a^2} \right) r^4$ . . . . $= -$ 0, 5465

Si valores nunc eruti in praecedenti aequatione ſubrogentur, deprehenditur

$h = 0,6918 - 0,3457 - 0,4007 - 0,5465$ &c. Quum autem ſeries haec manifeſto divergens ſit, nihilque proinde nos doceat, alia via in radicem aequationis tranſcendentis (M) inquirendum eſt.

10. Itaque per falſas poſitiones vadum tento; & pono primum $h = 30^0 = 0, 524$. Hinc derivatur

cos. h . . . . $=$ 0, 8660

log. a . . . . $=$ 0, 3174116

log. ſin. h . . . . $=$ 9, 6989700

og. a ſin. h . . . . $=$ 0, 0163816

$-$ *a ſin. h* . . . . $= -$ 1, 038

log. h . . . . $=$ 9, 7193313

$\log. h\, ſin. h$	$=$	$9,\ 4183017$
$-\ h\, ſin. h$	$=$	$-\ 0,\ 2620$
$\log. ſin. \lambda\, ſin. \delta$	$=$	$9,\ 4509883$
$\log. cos. \lambda\, cos. \delta$	$=$	$9,\ 8105987$
$\log. \frac{ſin. \lambda\, ſin. \delta}{cos. \lambda\, cos. \delta}$	$=$	$9,\ 6403898$
$\frac{ſin. \lambda\, ſin. \delta}{cos. \lambda\, cos. \delta}$	$=$	$0,\ 4369$

Igitur aequatio (M) fit

$0,\ 8660 - 1,\ 038 - 0,\ 2620 + 0,\ 4369 = 0,\ 0029$

adeoque error exceſſivus $= 0,\ 002$[illegible]

11. Sumo ſecundo $h = 45^{\circ} = 0,\ 785$; & conſequor

$cos. h$	$=$	$0,\ 7071$
$\log. ſin. h$	$=$	$9,\ 8494850$
$\log. a$	$=$	$0,\ 3174116$
$\log. a\, ſin. h$	$=$	$0,\ 1668966$
$-\ a\, ſin. h$	$=$	$-\ 1,\ 469$
$\log. h$	$=$	$9,\ 8948697$
$\log. h\, ſin. h$	$=$	$9,\ 7443547$
$-\ h\, ſin. h$	$=$	$-\ 0,\ 5551$

$\frac{\text{ſin.}\,\lambda\ \text{ſin.}\,\delta}{\text{cos.}\,\lambda\ \text{cos.}\,\delta}$ . . . . = 0, 4369

Ex hoc oritur aequatio (*M*)

0, 7071 — 1, 469 — 0, 5551 + 0, 4369 = − 0, 8801;
ubi error defectivus = — 0, 8801.

12. Quamobrem habebitur

Poſit. I. $h = 30^\circ = 0,\ 524$

Err. I. $= + 0,\ 0029$

Poſit. II. $h = 45^\circ = 0,\ 785$

Err. II. $= - 0,\ 8801$

Fiat jam uti Errorum ſumma ad Poſitionum differentiam ita Error minimus ad quartum proportionalem, ſeu

0, 8830: 15° : : 0, 0029 : 0, 04926° = 3′ circiter.

Quartus hic proportionalis additus primae Poſitioni efficit $h = 30^\circ\ 3' = 0,\ 5245$; habeoque

cos. h	. . . . =	0, 8656
log. a	. . . . =	0, 3174116
log. ſin. h	. . . . =	9, 6996258
log. a ſin. h	. . . . =	0, 0170374
— *a ſin. h*	. . . . =	− 1, 040
log. h	. . . . =	9, 7197190
log. h ſin. h	. . . . =	9, 4193448

$- h \; ſin.\, h$ . . . . $= - 0,\ 2626$

$\frac{ſin.\,\lambda \; ſin.\,\delta}{cos.\,\lambda \; cos.\,\delta}$ . . . . $= \ 0,\ 4369$

Proinde aequatio tranſcendens (M) abit in

$0,\ 8656 - 1,040 - 0,\ 2626 + 0,\ 4369 = - 0,\ 00..$

Defectivus Error ex hac Poſitione ortus evadit $- 0,\ 0001$.

13. Propterea ſit

$Poſit.\ I.\ \ h = 30^{\circ} = 0,\ 524$

$Err.\ I. = + 0,\ 0029$

$Poſit.\ III.\ \ h = 30^{\circ}.\ 3' = 0,\ 5245$

$Err.\ III. = - 0,\ 0001$

Inſtituatur rurſus analogia, quemadmodum eſt ſumma Errorum ad differentiam Poſitionum, ita Error minimus ad quartum, hoc eſt

$0,\ 003 : 3' :: 0,\ 0001 : 0,\ 1' = 6''$.

Si quartus hic proportionalis $6''$ a tertia Poſitione $30^{\circ}.\ 3'$ auferatur, invenitur ipſius h valor ſumme appropinquans nempe $h = 30^{\circ}.\ 3' - 6'' = 30^{\circ}.\ 2'.\ 54''$.

14. Facta modo anguli hujus horarii converſione in tempus prodeunt horae duae, & minuta ſecunda duodecim paullo minus poſt meridiem. Nemo autem ignorat, in noſtris hiſce regionibus ſolſtitialibus diebus aeſtum Solis maximum duabus circiter poſt meridiem horis ſaevire. Profecto tam admirabilis tamque inſpe-

ratus aequationis noſtrae cum veritate rerumque naturà conſenſus aperte docet quanta ſit in res phyſicas etiam abſtruſiores ſubtilioreſque Analyſeos rite adminiſtratae poteſtas & imperium.

P A R S II.

15. DEfinienda nunc ſupereſt aeſtiva dies, qua Solis calor maximum acquirit incrementum. Ratum habeo pluriumque illuſtrium Phyſicorum ſuffragio firmatum, calorem Solis die quovis aeſtivo ingruentem, cum calore diei cujuſvis alterius comparatum rationem ſequi compoſitam ex ratione directa ſimplici arcuum ſemidiurnorum, & ſinuum altitudinum Solis meridianarum, & ex duplicata inverſa diſtantiarum Solis a Terra, ita ut relativi caloris menſura aeſtivo quolibet die exprimatur per factum ex arcu ſemidiurno in ſinum altitudinis Solis meridianae diviſum per quadratum diſtantiae Solis a Terra in eodem die. Quum autem maximus Solis calor, in quem inquirimus, neque ante Solſtitium aeſtivum, ut palam eſt, neque poſt Æquinoctium autumnale incidere poſſit, & diſtantiae Solis

a Terra in hoc temporis intervallo mutabiles tam parum a ſe differant (*m*), ut caloris inde profecta variatio negligi tuto queat, quippe quae, ubi maxima ſit, ad tres partes centeſimas caloris ſolſtitialis non pertingit; propterea in indicata caloris formula contempto denominatore, ſeu diſtantiae quadrato numerator ſuperſtes caloris diurni vim ſufficienti accuratione repraeſentabit.

16. Si itaque altitudo Solis meridiana dato quolibet aeſtivo die dicatur A, arcus ſemidiurnus tunc temporis vocetur P, exponet formula P *ſin.* A calorem illius diei relativum. Conſtat porro ex Aſtronomiae rudimentis in Hemisphaerio noſtro Boreali aeſtivo tempore altitudinem Solis meridianam complementum eſſe arcus, qui metitur differentiam latitudinis loci & declinationis, ita nimirum ut ſit $A + \lambda - \delta = 90°$. Inde fiet *ſin.* $A = \cos. (\lambda - \delta)$, & formula P *ſin.* A abibit in aliam

$$P \cos. (\lambda - \delta) = P \cos. \lambda \cos. \delta + P\, ſin.\, \lambda\, ſin.\, \delta.$$

In hac hypotheſi caloris maximi accipiendum eſt formulae iſtius differentiale, & nihilo aequandum; ſed oportet prius duarum variabilium P, & δ alter-

(*m*) Vid. excellens Opus inſcriptum: *Recueil de Tables Aſtronomiques, publié ſous la direction de l'Académie Royale des Sciences & Belles Lettres de Pruſſe. Berlin* 1776, pag. 259. Vol. I.

utram eliminare. Ad hanc rem ſubſidium peto ab aequatione ſuperius (*Part. I.*) conſtituta

$$ſin.\,\varepsilon = cos.\,\lambda\; cos.\,\delta\; cos.\,h + ſin.\,\lambda\; ſin.\,\delta,$$

in qua ſi horarius angulus h pertingere ſumatur a meridie ad horizontem, evadit $h = P$, ſeu arcui ſemidiurno, & $\varepsilon = o$; adeoque aequatio fit $o = cos.\,\lambda \times cos.\,\delta\; cos.\,P + ſin.\,\lambda\; ſin.\,\delta$, & $cos.\,\lambda\; cos.\,\delta = -\frac{ſin.\,\lambda\, ſin\,\delta}{cos.\,P}$.

Hoc valore in caloris formula $P\,(cos.\lambda\; cos.\,\delta + ſin.\lambda\, ſin.\delta)$ ſubrogato oritur $P\left(ſin.\,\lambda\; ſin.\,\delta - \frac{ſin.\,\lambda\; ſin.\,\delta}{cos.\,P}\right)$

$$= \frac{P\, ſin.\,\lambda\; ſin.\,\delta}{cos.\,P}\,(cos.\,P - 1);$$

17. Ut nunc inveniatur valor $ſin.\,\delta$ expreſſus per functionem ipſius P, quadretur aequatio $cos.\,\lambda.\; cos.\,\delta = -\frac{ſin.\,\lambda\; ſin.\,\delta}{cos.\,P}$, unde eruitur $cos.^2\,\lambda\,(1 - ſin.^2\,\delta) = \frac{ſin.^2\,\lambda\; ſin.^2\,\delta}{cos.^2\,P}$, &

$$ſin.^2\,\delta = \frac{cos.^2\,\lambda\; cos.^2\,P}{ſin.^2\,\lambda + cos.^2\,\lambda\; cos.^2\,P};$$ eductaque radice

$$ſin.\,\delta = \frac{cos.\,\lambda\; cos.\,P}{\sqrt{(ſin.^2\,\lambda + cos.^2\,\lambda\, cos.^2\,P)}}.$$ Hunc valorem ſubſtituo in ſuperiori formula $\frac{P\, ſin.\,\lambda\; ſin.\,\delta}{cos.\,P}\,(cos.\,P - 1)$,

& habeo $\frac{P\, fin.\, \lambda\, cos.\, \lambda\, (cos.\, P - 1)}{\sqrt{(fin.^2 \lambda + cos.^2 \lambda\, cos.^2 P)}}$ aeſtus Solaris menſuram pro dato quolibet aeſtatis die. In hac poſtrema formula una tantum reperitur quantitas variabilis, nimirum P, quae in Boreali Hemiſphaerio aeſtivo tempore nonaginta gradus ſemper excedit, ideoque differentiale ipſius *cos.* P, quod ſecus foret negativum, in affirmativum vertitur. Hac itaque cautione ſumpto ejuſdem formulae differentiali invenio

$$[(d P\, cos.\, P\, fin.\, \lambda\, cos.\, \lambda + P\, d P\, fin.\, P\, fin.\, \lambda\, cos.\, \lambda)$$

$$\sqrt{(fin.^2 \lambda + cos.^2 \lambda\, cos.^2 P)} - \frac{P d P\, fin.\, P cos.^2 P fin. \lambda\, cos.^3 \lambda}{\sqrt{(fin.^2 \lambda + cos.^2 \lambda\, cos.^2 P)}}$$

$$- d P\, fin.\, \lambda\, cos.\, \lambda \sqrt{(fin.^2 \lambda + cos.^2 \lambda\, cos.^2 P)}$$

$$+ \frac{P d P\, fin.\, P\, cos.\, P fin.\, \lambda\, cos.^3 \lambda}{\sqrt{(fin.^2 \lambda + cos.^2 \lambda\, cos.^2 P)}}] : (fin.^2 \lambda + cos.^2 \lambda\, cos.^2 P) = 0,$$

& factis idoneis reductionibus tandem conſequor equationem (*N*)

$$P\, fin.\, P\, cos.\, \lambda\, fin.^3 \lambda + P\, fin.\, P\, cos.\, P\, fin.\, \lambda\, cos.^3 \lambda$$
$$+ cos.^3 P\, fin.\, \lambda\, cos.^3 \lambda - cos.^2 P\, fin.\, \lambda\, cos.^3 \lambda$$
$$+ cos.\, P\, cos.\, \lambda\, fin.^3 \lambda - cos.\, \lambda\, fin.^3 \lambda = 0$$

18. Æquatio tranſcendens adeo ſimplex, quae ſcilicet functionem complectitur duarum tantum quantitatum, in Problemate minime obvio & vulgari opta-

ri quidem, ſed ſperari nequaquam poterat. Hujus jam reſolutionem per falſas Poſitiones aggredior in anteceſſum animadvertens eſſe P $>90°$, ideoque *cos.* P ſemper negativum.

Pono itaque primo arcum ſemidiurnum P $= 7.\ 56. = 119° = 2,077$; eſt autem Ticini $\lambda = 45°.\ 11'$. En totius calculi typum.

$$P = 119° = 2,077$$

$$\lambda = 45°.\ 11'$$

log. P	$= 0,3174116$
log. ſin. P	$= 9,9418193$
log. cos. P	$= 9,6855712$
log. ſin. λ	$= 9,8508702$
log. cos. λ	$= 9,8480909$
log. ſin.^{3} λ	$= 9,5526106$
log. cos.^{3} λ	$= 9,5442727$
log. P *ſin.*P *cos.*λ *ſin.*^{3} λ	$= 9,6599324$
P *ſin.* P *cos.*λ *ſin.*^{3} λ	$= 0,457$
log. P *ſin.*P *cos.*P *ſin.*λ *cos.*^{3} λ . . .	$= 9,3399450$
P *ſin.* P *cos.*P *ſin.*λ *cos.*^{3} λ	$= -0,2187$
log. cos.^{3} P	$= 9,0567136$

$\log.\ \cos.^3 P\ \text{ſin}.\lambda\ \cos.^3\lambda$	$= 8,\ 4518567$
$\cos.^3 P\ \text{ſin}.\lambda\ \cos.^3\lambda$	$= -0,\ 0283$
$\log.\ \cos.^2 P$	$= 9,\ 3711424$
$\log.\ \cos.^2 P\ \text{ſin}.\lambda\ \cos.^3\lambda$	$= 8,\ 7662853$
$-\cos.^2 P\ \text{ſin}.\lambda\ \cos.^3\lambda$	$= -0,\ 05838$
$\log.\ \cos P\ \cos.\lambda\ \text{ſin}.^3\lambda$	$= 9,\ 0862727$
$\cos.\ P\ \cos.\lambda\ \text{ſin}.^3\lambda$	$= -0,\ 1221$
$\log.\ \cos.\lambda\ \text{ſin}.^3\lambda$	$= 9,\ 4007015$
$-\cos.\lambda\ \text{ſin}.^3\lambda$	$= -0,\ 2516$

Fit igitur aequatio (N)
$0,457 - 0,2187 - 0,0283 - 0,05838 - 0,1221 - 0,2516 = -0,22208$; ubi erratur per defectum $-0,22208$.

19. Aſſumo ſecundo $P = 90^\circ = 1,571$; in qua aſſumptione fit $\text{ſin}.\ P = 1$, $\cos.\ P = 0$, & formula (N) evadit

$$P\ \cos.\lambda\ \text{ſin}.^3\lambda - \cos.\lambda\ \text{ſin}.^3\lambda = 0.$$

Nunc autem eſt

$\log.\ P$	$= 0,\ 1961762$
$\log.\ P\ \cos.\lambda\ \text{ſin}.^3\lambda$	$= 9,\ 5968777$
$P\ \cos.\lambda\ \text{ſin}.^3\lambda$	$= 0,\ 3953$

$\frac{1}{2}$ *coſ.* λ *ſin.*[3] λ . . . . = − 0, 2516.

Igitur formula (*N*) invenitur

0, 3953 — 0, 2516 = 0, 1437

Quamobrem ex prima Poſitione Error naſcitur defectivus — 0, 22208, ex ſecunda exceſſivus + 0,1437, hoc eſt invenitur

Poſit. I. P = 119° = 2, 077

Err. I. = — 0, 22208

Poſit. II. P = 90° = 1, 571

Err. II. = + 0, 1437

Fiat modo uti Errorum ſumma ad Poſitionum differentiam, ita Error minimus ad quartum proportionalem, nimirum

0, 36578 : 29° :: 0, 1437 : 11, 39° = 11° 23′.

Quartus iſte proportionalis 11° 23′ additus Poſitioni ſecundae 90° dat P = 101° 23′ = 1, 769

20. Aſſumpto igitur tertio loco P = 101° 23′ = 1, 769 invenitur

log. P	. . . . =	0, 2478311
log. ſin. P	. . . . =	9, 9913717
log. coſ. λ *ſin.*[3] λ	. . . . =	9, 4007015
log. P *ſin.* P *coſ.* λ *ſin.*[3] λ	. . . . . =	9, 6399043
P *ſin.* P *coſ.* λ *ſin.*[3] λ	. . . . =	0, 4364

log. cos. P . . . . = 9, 2876[illegible]

log. P *ſin.* P *cos.* P *ſin.* λ *cos.*^{3} λ . . = 8, 9220[illegible]

P *ſin.* P *cos.* P *ſin.* λ *cos.*^{3} λ . . . =− 0, 0835[illegible]

log. cos.^{3} P . . . . = 7, 8630625

log. cos.^{3} P *ſin.* λ *cos.*^{3} λ . . . . . = 7, 2582054

cos.^{3} *P ſin.* λ *cos.*^{3} λ . . . . . . =− 0, 001812

log. cos.^{2} P . . . . = 8, 5753750

log. cos.^{2} P *ſin.* λ *cos.*^{3} λ . . . . = 7, 9705179

— *cos.*^{2} P *ſin.* λ *cos.*^{3} λ . . . . . =− 0, 009344

log. cos. P *cos.* λ *ſin.*^{3} λ . . . . . = 8, 6883890

cos. P *cos.* λ *ſin.*^{3} λ . . . . =− 0, 0488

— *cos.* λ *ſin.*^{3} λ . . . . =− 0, 2516

Hinc formula (*N*) convertitur in

0, 4364 — 0, 08357 — 0, 001812 — 0, 009344 — 0, 0488 — 0, 2516 = 0, 041274, prodeunte nimirum errore per exceſſum 0, 041274. Quare

Poſit. II. P = 90° = 1, 571

Err. II. = + 0, 1437

Poſit. III. P = 101°. 23′ = 1, 769

Err. III. = + 0, 041274

Nunc autem propter Errorum ſimilitudinem (nam

ambo ſunt per exceſſum) analogia inſtituitur hoc pacto, uti Errorum differentia ad differentiam Poſitionum, ita Error minimus ad quartum, hoc eſt

$0,\ 102426 : 683' :: 0,\ 041274 : 275,\ 2' = 4^{\circ}.\ 35'$.

Addatur porro quartus nunc erutus $4^{\circ}\ 35'$ Poſitioni tertiae $101^{\circ}\ 23'$, ut fiat $P = 105^{\circ}\ 58' = 1,\ 849$.

21. Pergo iterum, & pono quarto loco

$$P = 105^{\circ}\ 58' = 1,\ 849$$

log. P	= 0, 2670338
log. ſin. P	= 9, 9829140
log. cos. λ *ſin.*² λ	= 9, 4007015
log. P *ſin.* P *cos.* λ *ſin.*² λ	= 9, 6506493
P *ſin.* P *cos.* λ *ſin.*² λ	= 0, 4474
log. cos. P	= 9, 4394560
log. ſin. λ *cos.*² λ	= 9, 3951429
log. P *ſin.* P *cos.* P *ſin. cos.*² λ . .	= 9, 0845467
P *ſin.* P *cos.* P *ſin.* λ *cos.*² λ	= − 0, 1215
*log. cos.*³ P	= 8, 3183680
*log. cos.*³ P *ſin.* λ *cos.*² λ	= 7, 7135109
*cos.*³ P *ſin.* λ *cos.*² λ	= − 0, 00517
*log. cos.*² P	= 8, 8789120

$log.\ cos.^2 P\ ſin.\lambda\ cos.^3\lambda$ . . . . . $= 8,\ 27$[illegible]

$-\ cos.^2 P\ ſin.\lambda\ cos.^3\lambda$ . . . . . $= -0,\ 018$[illegible]

$log.\ cos.P\ cos.\lambda\ ſin.^3\lambda$ . . . . . $= 8,\ 840$[illegible]

$cos.P\ cos.\lambda\ ſin.^3\lambda$ . . . . $= -0,\ 06921$

$-\ cos.\lambda\ ſin.^3\lambda$ . . . . $= -0,\ 2516$

Conſequenter formula (*N*) abit in

0, 4474 — 0, 1215 — 0, 00517 — 0, 0188 — 0, 06921 — 0, 2516 = — 0, 01888; erratur ſcilicet per defectum — 0, 01888. Quamobrem

Poſit.III. P = 101.° 23′ = 1, 769

Err.III. = + 0, 041274

Poſit.IV. P = 105.° 58′ = 1, 849

Err.IV. = — 0, 01888

Inveniatur, ut antea, quartus proportionalis poſt Errorum ſummam, Poſitionum differentiam, & Errorem minimum, hoc eſt fiat

0, 060154 : 275′ : : 0, 01888 : 8, 631′ = 9′ circiter.

Tum quartus proportionalis modo inventus 9′ auferatur a Poſitione quarta, quo facto oritur P = 105.° 49′ = 1, 847.

22. Itaque pono demum quinto loco

P = 105.° 49′ = 1, 847

log. P	=	0, 2664186
log. ſin. P	=	9, 9832377
log. cos. λ ſin.^{3} λ	=	9, 4007015
log. P ſin. P cos. λ ſin.^{3} λ	=	9, 6503578
P ſin. P cos. λ ſin.^{3} λ	=	0, 4470
log. cos. P	=	9, 4354623
log. ſin. λ cos.^{3} λ	=	9, 3951429
log. P ſin. P cos. P ſin. λ cos.^{3} λ	=	9, 0802615
P ſin. P cos. P ſin. λ cos.^{3} λ	=	− 0, 1203
log. cos.^{3} P	=	8, 3063869
log. cos.^{3} P ſin. λ cos.^{3} λ	=	7, 7015298
cos.^{3} P ſin. λ cos.^{3} λ	=	− 0, 00503
log. cos.^{2} P	=	8, 8709246
log. cos.^{2} P ſin. λ cos.^{3} λ	=	8, 2660675
— cos.^{2} P ſin. λ cos.^{3} λ	=	− 0, 01845
log. cos. P cos. λ ſin.^{3} λ	=	8, 8361638
cos. P cos. λ ſin.^{3} λ	=	− 0, 06857
— cos. λ ſin.^{3} λ	=	− 0, 2516

Hinc formula (*N*) detegitur

0, 4470 — 0, 1203 — 0, 00503 — 0, 01845 — 0, 068[illegible]
— 0, 2516 = — 0, 01695. Iccirco habetur

Poſit. IV. P = 105.° 58′ = 1, 849

Err. IV. = — 0, 041274

Poſit. V. P = 105.° 49′ = 1, 847

Err. V. = — 0, 01695

Fiat iterum uti Errorum ſimilium differentia ad differentiam Poſitionum, ita Error minimus ad quartum proportionalem, nimirum

0, 024324 : 9′ : : 0, 01695 : 6, 286′ = 6′.

Hoc dempto a Poſitione quinta reſultat P = 105.° 43′ = 1, 845.

23. Pono rurſus ſexto loco

P = 105.° 43′ = 1, 845

log. P	. . . . =	0, 2660080
log. ſin. P	. . . . =	9, 9834517
log. cos. λ *ſin.*³ λ	. . . . =	9, 4007015
log. P *ſin.* P *cos.* λ *ſin.*³ λ	. . . . . =	9, 6501612
P *ſin.* P *cos.* λ *ſin.*³ λ	. . . . . . =	0, 4468
log. cos. P	. . . . =	9, 4327777
log. ſin. λ *cos.*³ λ	. . . . =	9, 3951429

$log.\ P\ ſin.P\ cos.P\ ſin.\lambda\ cos.^3\lambda$ . . = 9, 0773803

$P\ ſin.P\ cos.P\ ſin.\lambda\ cos.^3\lambda$ . . . =− 0, 1195

$log.\ cos.^3 P$ . . . . = 8, 2983331

$log.\ cos.^3 P\ ſin.\lambda\ cos.^3\lambda$ . . . . = 7, 6934760

$cos.^3 P\ ſin.\lambda\ cos.^3\lambda$ . . . . =− 0, 004937

$log.\ cos.^2 P$ . . . . = 8, 8655554

$log.\ cos.^2 P\ ſin.\lambda\ cos.^3\lambda$ . . . . . = 8, 2606983

— $cos.^2 P\ ſin.\lambda\ cos.^3\lambda$ . . . . . =− 0, 01823

$log.\ cos.P\ cos.\lambda\ ſin.^2\lambda$ . . . . . = 8, 8334792

$cos.P\ cos.\lambda\ ſin.^2\lambda$ . . . . =− 0, 06815

— $cos.\lambda\ ſin.^3\lambda$ . . . . =− 0, 2516

Hinc formula (*N*) invenitur
0, 4468 — 0, 1195 — 0, 004937 — 0, 01823 — 0,06815 — 0, 2516 = — 0, 015617. Iccirco habetur

Poſit. V. P = 105.° 49′ = 1, 847

Err. V. = — 0, 01695

Poſit. VI. P = 105.° 43′ = 1, 845

Err. VI. = — 0, 015617

Fiat rurſus uti Errorum ſimilium differentia ad differentiam Poſitionum, ita Error minimus ad quartum hoc eſt

0, 001333 : 6′ : : 0, 015617 : 70, 29′ = 1.° 10′.

Quartus proportionalis 1° $10'$ ablatus a Poſitione ſexta 105° $43'$ praebet $P = 104^\circ\ 33' = 1,825$.

24. Pono igitur ſeptimo loco

$$P = 104^\circ\ 33' = 1,825$$

log. P	=	0, 2611886
log. ſin. P	=	9, 9858434
log. cos. P	=	9, 4000625
log. $cos.^2$ P	=	8, 8001250
log. $cos.^3$ P	=	8, 2001876
log. cos. λ $ſin.^3$ λ	=	9, 4007015
log. ſin. λ $cos.^3$ λ	=	9, 3951429
log. P *ſin.* P *cos* λ $ſin.^3$ λ	=	9, 6477335
P *ſin.* P *cos.* λ $ſin.^3$ λ	=	0, 4444
log. P *ſin.* P *cos.* P *ſin.* λ $cos.^3$ λ	=	9, 0422374
P *ſin.* P *cos.* P *ſin.* λ $cos.^3$ λ	=	− 0, 1102
log. $cos.^3$ P *ſin.* λ $cos.^3$ λ	=	7, 5953304
$cos.^3$ P *ſin.* λ $cos.^3$ λ	=	− 0, 003938
log. $cos.^2$ P *ſin.* λ $cos.^3$ λ	=	8, 1952679
— $cos.^2$ P *ſin.* λ $cos.^3$ λ	=	− 0, 01568
log. cos. P *cos.* λ $ſin.^3$ λ	=	8, 8007640

$\cos. P \cos. \lambda \, \text{ſin}.^3 \lambda$ . . . . $= - 0,\ 06321$

$- \cos.^2 \lambda \, \text{ſin}.^3 \lambda$ . . . . $= - 0,\ 2516$

Quapropter aequatio (*N*) fit

$0,4444 - 0,1102 - 0,003938 - 0,01568 - 0,06321 - 0,2516 = - 0,0002$. Eſt igitur

Poſit. VI. $P = 105^{\circ}.\ 43' = 1,\ 845$

Err. VI. $= - 0,\ 015617$

Poſit. VII. $P = 104^{\circ}.\ 33' = 1,\ 825$

Err. VII. $= - 0,\ 0002$

Inſtituatur rurſus analogia, uti Errorum differentia ad differentiam Poſitionum, ita Error minimus ad quartum, videlicet $0,\ 0154 : 70' :: 0,0002 : 0,\ 9092' = 54''$. Auferatur jam a Poſitione ſeptima $104^{\circ}.\ 33'$ quartus modo inventus $54''$, & orietur tandem aequationis (*N*) radix vero maxime appropinquans $P = 104^{\circ}.\ 32'.\ 6''$.

25 Igitur invenimus demum arcum ſemidiurnum calori aeſtivo vehementiſſimo reſpondentem, qui porro arcus ſi convertatur in tempus praebet $6^{h}.\ 58^{m}.$, nimirum horas ſex, & minuta quinquaginta octo. Invento autem arcu ſemidiurno facile erit declinationem Solis, proindeque & diem quaeſitam detegere. Enimvero nacti antea ſumus aequalitatem

$$\text{ſin}.\,\delta = \frac{\cos.\lambda\ \cos.P}{\sqrt{(\text{ſin}.^2\lambda + \cos.^2\lambda\ \cos.^2P)}}.$$ Propterea

ſubducto calculo fiet

$$P = 104.^\circ\ 32'$$

$\log.\ \cos.\lambda$	=	9, 8480909
$\log.\ \cos.P$	=	9, 3995754
$\log.\ \cos.\lambda\ \cos.P$	=	9, 2476663
$\log.\ \text{ſin}.\lambda$	=	9, 8508702
$\log.\ \text{ſin}.^2\lambda$	=	9, 7017404
$\text{ſin}.^2\lambda$	=	0, 5032
$\log.\ \cos.^2\lambda$	=	9, 6961818
$\log.\ \cos.^2P$	=	8, 7991508
$\log.\ \cos.^2\lambda\ \cos.^2P$	=	8, 4953326
$\cos.^2\lambda\ \cos.^2P$	=	0, 03128
$\text{ſin}.^2\lambda + \cos.^2\lambda\ \cos.^2P$	=	0, 53448
$\log.(\text{ſin}.^2\lambda + \cos.^2\lambda\ \cos.^2P)$ . .	=	9, 7279315
$\log.\sqrt{(\text{ſin}.^2\lambda + \cos.^2\lambda\ \cos.^2P)}$ . .	=	9, 8639657
$\log.\frac{\cos.\lambda\ \cos.P}{\sqrt{(\text{ſin}.^2\lambda + \cos.^2\lambda\cos.^2P)}} = \log.\text{ſin}.\delta$	=	9, 3837006

Hinc eruitur $\delta = 14.^{\circ}$, quod incidit in diem $14.^{um}$ & $15.^{um}$ menſis auguſti. Is itaque dies prae aeſtivis omnibus Maximo Caloris gradu infamis erit. Hoc vero ſi cum Phyſicorum obſervationibus calorem maximum in noſtro climate circiter finem Julii, & initium Auguſti indicantibus non quadrat ad amuſſim, tam parum tamen ab illis diſcordat, ut in quaeſtione tam lubrica tantiſque impedita difficultatibus majorem conſenſum nemo poſtulaverit niſi totius rei phyſicae, & mathematicae rudis.

26. Experiamur nunc ſtrictim curſimque, quid in aeſtimatione Caloris Annui Maximi obveniat, ſi loco arcus ſemidiurni ſimplicis adhibeatur quadratum illius, ut non paucis Phyſicis placet. In hac itaque hypotheſi rationes repetens §. 16. ſubductas invenio calorem diei cujuſvis aeſtivi ab expreſſione

$$\frac{P^2 \; \sin.\lambda \; \cos.\lambda \; (\cos.P - 1)}{\sqrt{(\sin.^2\lambda + \cos.^2\lambda \; \cos.^2 P)}}$$ repraeſentatum. In hac expreſſione variat ſola quantitas P, eſtque ea aeſtatis tempore ſemper major arcu quadrantali, indeque negativus eſt $\cos.$ P, & viciſſim affirmativum differentiale ejuſdem. Quamobrem ſi hac circumſpectione accipiatur caloris modo inventi differentiale, illudque nihilo aequetur, per idonea calculi compendia ad ſe-

quentem demum pervenimus aequationem

P^2 *ſin.* P *cos.* λ *ſin.*^{3} λ + P^2 *ſin.* P *cos.* P *ſin.* λ *cos.*^{3} λ
+ 2P *cos.* P *cos.* λ *ſin.*^{3} λ + 2P *cos.*^{2} P *ſin.* λ *cos.*^{3} λ
− 2P *cos.*^{2} P *ſin.* λ *cos.*^{3} λ − 2P *cos.* λ *ſin.*^{3} λ = 0.

27. Capto, ut ante, λ = 45.° 11′, fiat P = 110° = 1, 9199; videamuſque quemnam valorem ex hiſce aſſumptionibus ſuperior aequatio nanciſcatur. Igitur

P = 110° = 1, 9199.

λ = 45.° 11′.

log. P	= 0, 2832786
log. P^2	= 0, 5665572
log. ſin. P	= 9, 9729858
log. cos. λ *ſin.*^{3} λ	= 9, 4007015
log. P^2 *ſin.* P *cos.* λ *ſin.*^{3} λ	= 9, 9402445
P^2 *ſin.* P *cos.* λ *ſin.*^{3} λ	= 0, 8715
log. cos. P	= 9, 5340517
log. P^2 *ſin.* P *cos.* P	= 0, 0735947
log. ſin. λ *cos.*^{3} λ	= 9, 3951429
log. P^2 *ſin.* P *cos.* P *ſin.* λ *cos.*^{3} λ . .	= 9, 4687376
P^2 *ſin.* P *cos.* P *ſin.* λ *cos.*^{3} λ . . .	= − 0, 2943
log. 2	= 0, 3010300

log. P	=	0, 2832786
log. cos. P	=	9, 5340517
log. cos. λ *ſin.*^{3} λ	=	9, 4007015
log. 2 P *cos.* P *cos.* λ *ſin.*^{3} λ	=	9, 5190618
2 P *cos.* P *cos.* λ *ſin.*^{3} λ	=	− 0, 3304
log. 2 P	=	6, 5843086
log. cos.^{3} P	=	8, 6021551
log. ſin. λ *cos.*^{3} λ	=	9, 3951429
log. 2 P *cos.*^{3} P *ſin.* λ *cos.*^{3} λ	=	8, 5816066
2 P *cos.*^{3} P *ſin.* λ *cos.*^{3} λ	=	− 0, 0382
log. 2 P	=	0, 5843086
log. cos.^{2} P	=	9, 0681034
log. ſin. λ *cos.*^{3} λ	=	9, 3951429
log. 2 P *cos.*^{2} P *ſin.* λ *cos.*^{3} λ	=	9, 0475549
— 2 P *cos.*^{2} P *ſin.* λ *cos.*^{3} λ	=	− 0, 1116
log. 2 P	=	0, 5843086
log. cos. λ *ſin.*^{3} λ	=	9, 4007015
log. 2 P *cos.* λ *ſin.*^{3} λ	=	9, 9850101
— 2 P *cos.* λ *ſin.*^{3} λ	=	− 0, 9661

Quamobrem ſuperior aequatio abit in 0,8715 — 0,29
— 0,3304 — 0,0382 — 0,1116 — 0,9661 = — 0,86
Hinc erratur per defectum quantitate — 0, 8691.

28. Sumo ſecundo loco

$$P = 105^\circ = 1,8326$$

$$\lambda = 45^\circ\ 11'$$

log. P	= 0, 2630677
log. P^2	= 0, 5261354
*log. ſin.*P	= 9, 9849438
log. cos. λ *ſin.*^{3} λ	= 9, 4007015
log. P^2 *ſin.*P *cos.* λ *ſin.*^{3} λ	= 9, 9117807
P^2 *ſin.*P *cos.* λ *ſin.*^{3} λ	= 0, 8162
log. P^2 *ſin.*P	= 0, 5110792
*log. cos.*P	= 9, 4129962
log. ſin. λ *cos.*^{3} λ	= 9, 3951429
log. P^2 *ſin.*P *cos.*P *ſin.* λ *cos.*^{3} λ . .	= 9, 3192183
P^2 *ſin.*P *cos.*P *ſin.* λ *cos.*^{3} λ	= - 0, 2085
log. 2	= 0, 3010300
log. P	= 0, 2630677
*log. cos.*P	= 9, 4129962

$\log.\ \cos.\ \lambda\ \sin.^3\ \lambda$	$= 9,\ 4007015$
$\log.\ 2P\ \cos.\ P\ \cos.\ \lambda\ \sin.^3\ \lambda$	$= 9,\ 3777954$
$2P\ \cos.\ P\ \cos.\ \lambda\ \sin.^3\ \lambda$	$= -0,\ 2387$
$\log.\ 2P$	$= 0,\ 5640977$
$\log.\ \cos.^3\ P$	$= 8,\ 2389886$
$\log.\ \sin.\ \lambda\ \cos.^3\ \lambda$	$= 9,\ 3951429$
$\log.\ 2P\ \cos.^3\ P\ \sin.\ \lambda\ \cos.^3\ \lambda$	$= 8,\ 1982292$
$2P\ \cos.^3\ P\ \sin.\ \lambda\ \cos.^3\ \lambda$	$= -0,\ 0158$
$\log.\ 2P$	$= 0,\ 5640977$
$\log.\ \cos.^2\ P$	$= 8,\ 8259924$
$\log.\ \sin.\ \lambda\ \cos.^3\ \lambda$	$= 9,\ 3951429$
$\log.\ 2P\ \cos.^2\ P\ \sin.\ \lambda\ \cos.^3\ \lambda$	$= 8,\ 7852330$
$-\ 2P\ \cos.^2\ P\ \sin.\ \lambda\ \cos.^3\ \lambda$	$= -0,\ 0610$
$\log.\ 2P$	$= 0,\ 5640977$
$\log.\ \cos.\ \lambda'\ \sin.^3\ \lambda$	$= 9,\ 4007015$
$\log.\ 2P\ \cos.\ \lambda\ \sin.^3\ \lambda$	$= 9,\ 9647992$
$-\ 2P\ \cos.\ \lambda\ \sin.^3\ \lambda$	$= -0,\ 9221$

Hinc praecedens aequatio ſequentem nanciſcitur valorem 0, 8162 — 0, 2085 — 0, 2387 — 0, 0158.

— 0°, 0610 — 0, 9221 = — 0, 6299, hoc eſt peccat iterum defectu.

29. Accipiatur nunc arcus ſemidiurnus ipſius diei ſolſtitialis aeſtivi, quando ſcilicet $\delta = 23° 28'$. Id aſſequemur per formulam §. 25. $\mathit{ſin}.\,\delta = \frac{\cos.\lambda \cos.P}{\sqrt{(\mathit{ſin}.^2\lambda + \cos.^2\lambda \cos.^2 P)}}$, ex qua opportunis peractis reductionibus eruitur $\cos.P = \mathit{tang}.\,\delta\ \mathit{tang}.\,\lambda$. Igitur

log. tang. δ	. . . . =	9, 6376106
log. tang. λ = *log. tang.* 45° 11'	. =	10, 0027793
log. tang. δ *tang.* λ	. . . . =	9, 6403899

Quare ſi P eſſet quadrante minor prodiret *cos.* P = *cos.* 64° 6'; quum autem, ut alias conſtat, ſit P quadrante major fit ideo *cos.* P = *cos.* (180° − 64° 6') = *cos.* 115° 54', ſeu P = 115° 54'. Ponatur itaque tertio loco

$$P = 115° 54' = 2, 0228$$

$$\lambda = 45° 11'$$

log. P	. . . . =	0, 3059529
log. P^2	. . . . =	0, 6119058
log. ſin. P	. . . . =	9, 9540291
log. cos. λ *ſin.*^{2} λ	. . . . =	9, 4007015

. P^2 *ſin.* P *cos.* λ *ſin.*^{2} λ . . .	=	0, 2725893
P^2 *ſin.* P *cos.* λ *ſin.*^{2} λ	=	1, 8732
log. cos. P	=	9, 6402844
log. P^2 *ſin.* P	=	0, 5659349
log. ſin. λ *cos.*^{2} λ	=	9, 3951429
log. P^2 *ſin.* P *cos.* P *ſin.* λ *cos.*^{2} λ .	=	9, 6013622
P^2 *ſin.* P *cos.* P *ſin.* λ *cos.*^{2} λ	=	− 0, 3994
log. 2	=	0, 3010300
log. P	=	0, 3059529
log. cos. P	=	9, 6402844
log. cos. λ *ſin.*^{2} λ	=	9, 4007015
log. 2 P *cos.* P *cos.* λ *ſin.*^{2} λ . . .	=	9, 6479688
2 P *cos.* P *cos.* λ *ſin.*^{2} λ	=	− 0, 4446
log. 2 P	=	0, 6069829
log. cos.^{2} P	=	8, 9208632
log. ſin. λ *cos.*^{2} λ	=	9, 3951429
log. 2 P *cos.*^{2} P *ſin.* λ *cos.*^{2} λ	=	8, 9229790
2 P *cos.*^{2} P *ſin.* λ *cos.*^{2} λ	=	− 0, 0837
log. 2 P	=	0, 6069829

$log.\ cos.^2 P$	. . . . =	9, 2805688
$log.\ ſin.\ \lambda\ cos.^3 \lambda$	. . . . =	9, 3951429
$log.\ 2 P\ cos.^2 P\ ſin.\ \lambda\ cos.^3 \lambda$	. . . =	9, 2826946
$-\ 2 P\ cos.^2 P\ ſin.\ \lambda\ cos.^3 \lambda$	. . . . =	$-$ 0, 1917
$log.\ 2 P$	. . . . =	0, 6069829
$log.\ cos.\ \lambda\ ſin.^3 \lambda$	. . . . =	9, 4007015
$log.\ 2 P\ cos.\ \lambda\ ſin.^3 \lambda$	. . . . =	0, 0076844
$-\ 2 P\ cos.\ \lambda\ ſin.^3 \lambda$	. . . . =	$-$ 1, 1018

Propterea ſuperior aequatio hanc induit formam 1, 8732 — 0, 3994 — 0, 4446 — 0, 0837 — 0, 1917 — 0, 1018 = — 0, 3480, aberrat ſcilicet a nihilo per defectum — 0, 3480.

30. Adhibeatur poſtremo P = 90°; tuncque ob *cos*.P = 0, *ſin*.P = 1 aequatio praecedens ad duos tantum terminos contrahitur $P^2 cos.\ \lambda\ ſin^3 \lambda - 2 P\ cos.\ \lambda\ ſin.^3 \lambda = 0$. Eſt igitur

$$P = 90^\circ = 1,5708$$
$$\lambda = 45^\circ.\ 11'$$

$log.\ P$	. . . . =	0, 1961209

$log.\ P^2$	. . . .	=	0, 3922418
$log.\ cos.\lambda\ ſin.^3\ \lambda$	. . . .	=	9, 4007015
$log.\ P^2\ cos.\lambda\ ſin.^3\ \lambda$	. . . . . .	=	9, 7929433
$P^2\ cos.\lambda\ ſin.^3\ \lambda$	. . . .	=	0, 6208
$log.\ 2$	. . . .	=	0, 3010300
$log.\ P$	. . . .	=	0, 1961209
$log.\ cos.\lambda\ ſin.^3\ \lambda$	. . . .	=	9, 4007015
$log.\ 2P\ cos.\lambda\ ſin.^3\ \lambda$	. . . . . .	=	9, 8978524
$-\ 2P\ cos.\lambda\ ſin.^3\ \lambda$	. . . . . .	=	− 0, 7904

Itaque ſuſcipit aequatio formam 0, 6208 — 0, 7904 = — 0, 1696, ſeu tantundem aberrat a genuino valore, nimirum a nihilo.

31. Ex dictis ſequens habetur aſſumptionum, errorumque conſpectus:

Arcus Semidiurni	Errores
115.° 54'	— 0, 3480
110.°	— 0, 8691
105.°	— 0, 6299
90.°	— 0, 1696

quem qui paullo attentius examinaverit, Errorumque indolem, menſuram, progreſſum, atque illorum cum Aſſumptionibus reſpondentibus relationes rite perpenderit, is apodictice invicteque colliget, Calorem Maximum, quem hic unice ſpectamus, in nullum incidere poſſe tempus, quod cum Phyſicorum obſervationibus vel aliquantulum conſpiret, & conſequenter hypotheſim de quadrato arcus ſemidiurni ante dictam excuſſis cardinibus convulſiſque fundamentis corruere.



Tab. I.

Fig. 1.

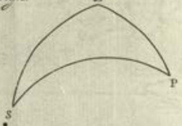


Fig. 2.

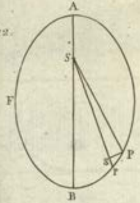


Fig. 3.

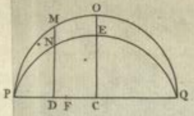


Fig. 4.

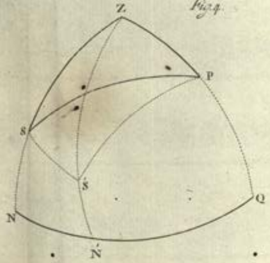
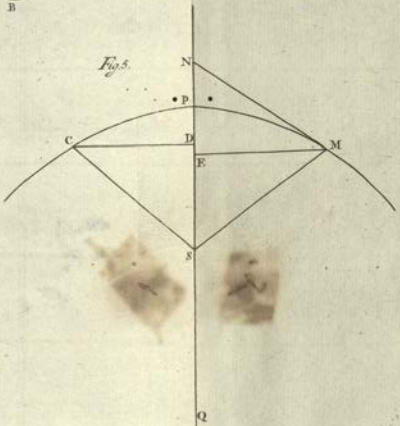


Fig. 5.



Tab. II.

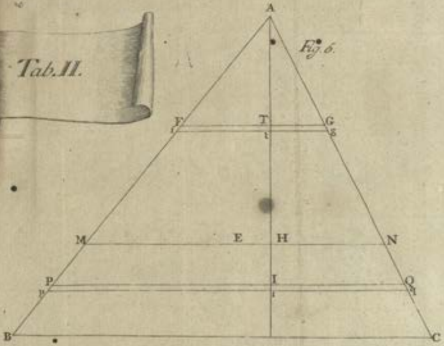


Fig. 6.

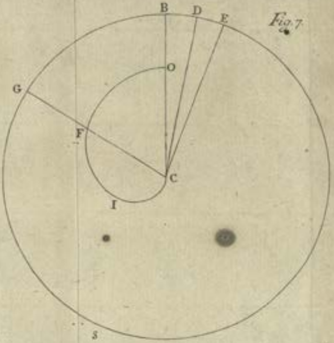


Fig. 7.

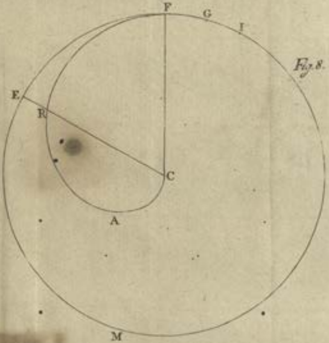


Fig. 8.

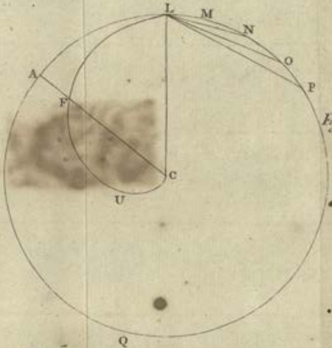


Fig. 9.

Tab. III

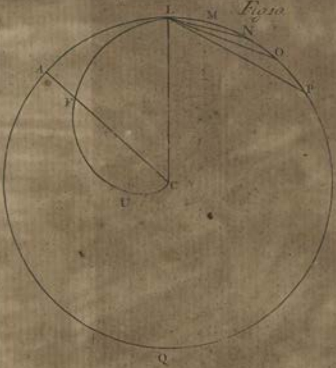


Fig. 10.

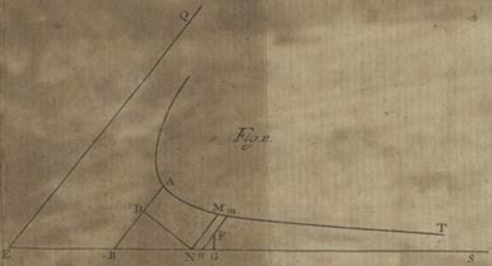


Fig. 11.

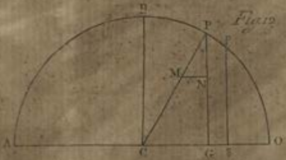


Fig. 12.

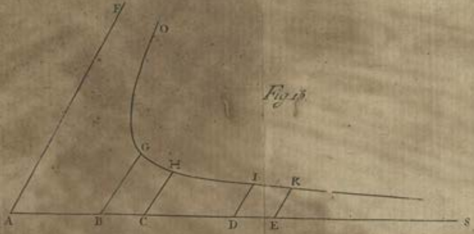


Fig. 13.

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